

# Review of Final Draft Core Standards

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What follows are my comments on the final draft of the CCSSI Core Mathematics Standards. There are a number of standards including, but not limited to 1-OA(6), 2-OA(2), 2-NBT(5), 3-OA(7), 3-NBT(2), 4-OA(4), 4-OA(6), 4-NF(1), 4-NF(2), 5-OA(3), 8-G(2), 8-G(4), F-LQE(5), G-SRT(4) that are completely unique to this document, and most of them seem problematic to me. I have repeatedly asked for references justifying the insertions of these or similar standards in previous drafts, but references have not been provided. Consequently, to my knowledge, there is no real research base for including any of these standards in the document.

## Basic Arithmetic and Arithmetic Operations.

Here are 1-OA(6), 2-OA(2), 3-OA(7), 2-NBT(5), 3-NBT(2), 3-OA(4), and 3-OA(6):

1.OA(6) Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g.,  $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$ ); decomposing a number leading to a ten (e.g.,  $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$ ); using the relationship between addition and subtraction (e.g., knowing that  $8 + 4 = 12$ , one knows  $12 - 8 = 4$ ); and creating equivalent but easier or known sums (e.g., adding  $6 + 7$  by creating the known equivalent  $6 + 6 + 1 = 12 + 1 = 13$ ).

2.OA(2) Fluently add and subtract within 20 using mental strategies.

2 By end of Grade 2, know from memory all sums of two one-digit numbers.

2-NBT(5) Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

3-NBT(2) Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

3-OA(7) Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that  $8 \times 5 = 40$ , one knows  $40 \div 5 = 8$ ) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

3-OA(4) . Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations  $8 \times ? = 48$ ,  $5 = ? \div 3$ ,  $6 \times 6 = ?$ .

3-OA(6) . Understand division as an unknown-factor problem. For example, find  $32 \div 8$  by

finding the number that makes 32 when multiplied by 8.

Note that

- most of these standards have some sort of fluency requirement for operations in a range, but no requirement that the algorithm being used is either general or generalizable. Also,
- note the extremely excessive pedagogical constraints in 1-OA(6), 3-OA(7).
- Note that 3-OA(6) is actually a definition, and part of a definition that is given at least one year earlier in virtually all the high achieving countries at that.

Specifically, subtraction is defined in the following way:  $a - b$  is that number  $c$ , if it exists, so that  $b + c = a$ , while division is defined by  $a \div b$  is that number,  $c$ , if it exists, so that  $b \times c = a$ .

With these understandings, the students in the high achieving countries only have to learn and master two operations, addition and multiplication, since the other two come along for free. Moreover, this is a key piece of the underpinnings for their success. But we are, instead, given 3-OA(6) which is neither fish nor fowl.

As regards fluency, I note that ultimately with

4-NBT(4) Fluently add and subtract multi-digit whole numbers using the standard algorithm.

5-NBT(5) Fluently multiply multi-digit whole numbers using the standard algorithm.

6-NS(2) Fluently divide multi-digit numbers using the standard algorithm.

expectations are that students will fluently operate with reasonable variants of the standard algorithms. But what will be the effects of the previous fluency requirements, except long-term confusion about key details of what is to be expected? So we can well imagine average and weaker students using some weird mnemonics to handle operations in certain ranges, and trying to combine this with a kind of dim understanding of how the standard algorithms work.

To further add to the confusion surrounding these core standards, note the following entirely reasonable standards that can only be regarded as *competing with the “fluency” standards within the document*.

2-NBT(6) Add up to four two-digit numbers using strategies based on place value and properties of operations.

2-NBT(7) Add and subtract within 1000, using concrete models or drawings and strategies based

on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

- 2-NBT(9) Explain why addition and subtraction strategies work, using place value and the properties of operations *This is almost certainly too advanced for second grade, but indicates a viable direction for student exploration in this and later grades.*
- 3-NBT(3) Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g.,  $9 \times 80$ ,  $5 \times 60$ ) using strategies based on place value and properties of operations.
- 4-NBT(5) Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
- 4-NBT(6) Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
- 5-NBT(6) Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

The seven standards above would have been exemplary if they had not occurred after the “fluency” standards for unconstrained algorithms that I had objected to at the beginning of this discussion. Within the document itself, there seems to be a minor war going on, and this is not something we should hand over to our teachers.

The above standards illustrate many serious flaws in the Core Standards. Also among these difficulties are that a large number of the arithmetic and operations, as well as the place value standards are one, two or even more years behind the corresponding standards for many if not all the high achieving countries. Consequently, I was not able to certify that the Core Mathematics Standards are benchmarked at the same level as the standards of the high achieving countries in mathematics.

## FRACTIONS

Just as there are serious concerns with the coherence of the Core Standards for basic arithmetic and place-value, there are also concerns with the coherence of the Core Standards for fractions, though here the difficulties are somewhat more subtle. Fractions first appear in grade 3 in Core Standards which is somewhat late by international expectations, but not too out of line.

- 3-NF(1) Understand a fraction  $1/b$  as the quantity formed by 1 part when a whole is partitioned into  $b$  equal parts; understand a fraction  $a/b$  as the quantity formed by  $a$  parts of size  $1/b$ .
- 3-NF(2) Understand a fraction as a number on the number line; represent fractions on a number line diagram.
- Represent a fraction  $1/b$  on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into  $b$  equal parts. Recognize that each part has size  $1/b$  and that the endpoint of the part based at 0 locates the number  $1/b$  on the number line.
  - Represent a fraction  $a/b$  on a number line diagram by marking off  $a$  lengths  $1/b$  from 0. Recognize that the resulting interval has size  $a/b$  and that its endpoint locates the number  $a/b$  on the number line.
- 3-NF(3) Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
- Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
  - Recognize and generate simple equivalent fractions, e.g.,  $1/2 = 2/4$ ,  $4/6 = 2/3$ . Explain why the fractions are equivalent, e.g., by using a visual fraction model.
  - Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. Examples: Express 3 in the form  $3 = 3/1$ ; recognize that  $6/1 = 6$ ; locate  $4/4$  and 1 at the same point of a number line diagram.
  - Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols  $>$ ,  $=$ , or  $<$ , and justify the conclusions, e.g., by using a visual fraction model.

**Remark.** This “visual fraction model” represents all that is wrong in our standard approach to fractions - an approach that has seldom worked. From the glossary, we have

VISUAL FRACTION MODEL. A tape diagram, number line diagram, or area model.

In short, what is done is to use three separate and basically unconnected models for fractions to decide if statements are true, false, or ambiguous. In particular, referring back to 3-NF(3), we have the separate notions of *position on the number line* and *size*. These are initially very different concepts when matched to student experience with numbers. Size refers to counting, but when dealing with fractions, counting is not appropriate except in the “partitive” model, which is abandoned very early in the development of this subject in the high achieving countries.

Indeed, what is done in the high achieving countries is to refer fractions entirely back to the number-line as soon as this becomes feasible – usually sometime in second grade or at the beginning of grade 3 – and not refer to size except in-so-far as a number on the number line to the right of another number is said to be larger.

Another point where the handling of fractions is problematic is in fourth grade:

- 4-NF(1) Explain why a fraction  $a/b$  is equivalent to a fraction  $(n \times a)/(n \times b)$  by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

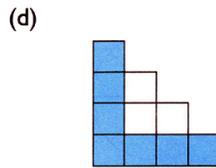
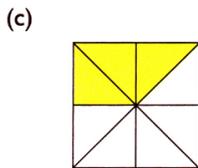
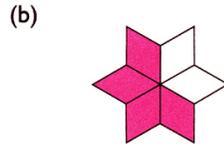
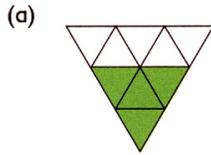
There are many ways to handle this, but visual fraction models is pretty much the worst. One thing that can be done is to observe that the *point on the number line* associated to  $(n \times a)/(n \times b)$  is exactly the same as the point associated to  $a/b$  provided  $b \neq 0$ . So “equivalent fraction” can be taken to mean “fraction representation by the same point on the number line,” and, again, in the high achieving countries, this is the approach taken:

For information, here is the teaching sequence in grades 2 - 4 for fractions in Singapore:

**Teaching sequence, Singapore, grades 2 - 4** The initial presentation of fractions in the Singapore programs occurs in the second half of second grade, and is developed using an area model where care is taken to be sure that the regions that decompose a geometric

figure are have the same area:

4. What fraction of each shape is coloured?



7. (a)



$\frac{3}{5}$  and  $\frac{2}{5}$  make 1 whole.

(b)

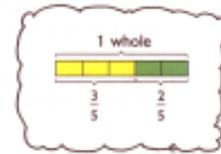


$\frac{1}{7}$  and  $\frac{6}{7}$  make 1 whole.

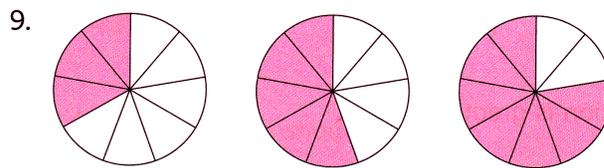
(c)



$\frac{2}{9}$  and  $\frac{7}{9}$  make 1 whole.



In the second part of the grade three text fractions continue to be developed using an area model, but the level of sophistication as increased significantly:

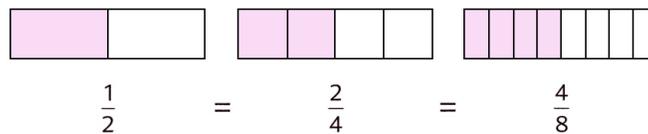


The fractions  $\frac{3}{9}$ ,  $\frac{5}{9}$  and  $\frac{7}{9}$  have a common denominator.

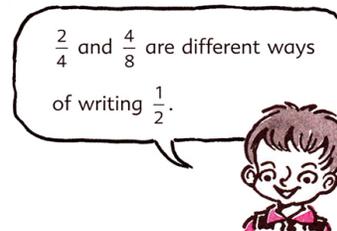
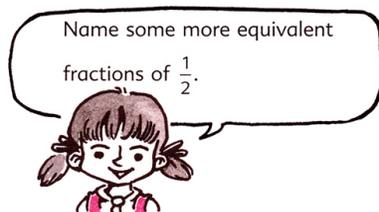
is the smallest fraction.

is the greatest fraction.

and equaivalent fractions are introduced

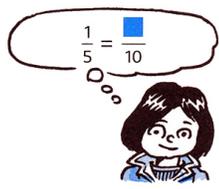
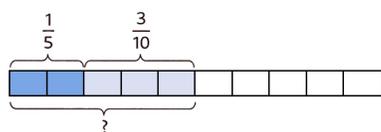


$\frac{1}{2}$ ,  $\frac{2}{4}$  and  $\frac{4}{8}$  are **equivalent fractions**.

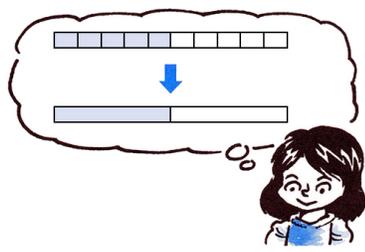


In the fourth grade the area model is moved systematically towards seeing fractions on the number line as the basic operations of addition and subtraction of fractions are developed:

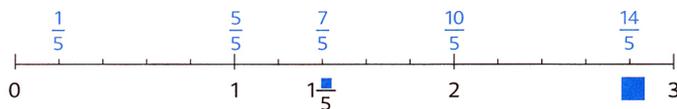
7. Add  $\frac{1}{5}$  and  $\frac{3}{10}$ .



$$\begin{aligned} \frac{1}{5} + \frac{3}{10} &= \frac{2}{10} + \frac{3}{10} \\ &= \frac{5}{10} \\ &= \frac{1}{2} \end{aligned}$$



3. Change the improper fractions to mixed numbers.



$$\begin{aligned} \text{(a)} \quad \frac{7}{5} &= \frac{5}{5} + \frac{2}{5} \\ &= 1 + \frac{2}{5} \\ &= 1\frac{2}{5} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{14}{5} &= \frac{10}{5} + \frac{4}{5} \\ &= 2 + \frac{4}{5} \\ &= 2\frac{4}{5} \end{aligned}$$

**Remark:** In the Russian texts translated by UCSMP the sequencing is very similar except that representing fractions on the number line is already present in grade 3.

The next problem is with the standard

4-NF(2) Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as  $\frac{1}{2}$ . Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols  $>$ ,  $=$ , or  $<$ , and justify the conclusions, e.g., by using a visual fraction model.

The first part of this standard is exemplary, but it is completely distorted by what follows. What does it mean to compare to “a benchmark fraction?” And this is only made worse by the requirement that students “recognize that comparisons are valid only when two fractions refer to the same whole.” This is an entirely unappetizing admixture of apples and spoiled oranges.

## Geometry

The approach to geometry in Core Standards is very unusual, focusing in eighth grade and beyond on using the Euclidian and extended Euclidean groups to define congruence and similarity as well as develop their key properties. Mathematically, this approach is rigorous, but it would generally be regarded as something that would be done in a college level geometry course for math majors. The exposition at the high school level seems plausible, and may well work. However, to my knowledge, there is no solid research that justifies this approach at the K-12 level currently.

It is also worth noting that a similar approach was taken in Russia about 30 years back, but was quickly rejected. It wasn't that the teachers were not capable of teaching, though this may well be a problem for most middle school and many high school math teachers here. The problem was that it was way too non-standard, and basic geometric facts and theorems had to be handled in entirely new, untested, and ultimately unsuccessful ways.

Here are some details on the issues that arise in geometry.

- 3-MD(5) Recognize area as an attribute of plane figures and understand concepts of area measurement.
- a. A square with side length 1 unit, called “a unit square,” is said to have “one square unit” of area, and can be used to measure area.
  - b. A plane figure which can be covered without gaps or overlaps by  $n$  unit squares is said to have an area of  $n$  square units.

Of course, the basic issue is that most figures in the plane are not decomposed into  $n$  unit squares without gaps or overlaps. For example, what of the triangle? 3-MD(5) is a good *beginning* for the discussion of area, but it is not more than this.

In fourth grade we have

- 4-MD(1) Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...

This is the summative standard for a whole sequence of standards that start in the earliest grades but continue through grade 5 or even grade 6. It is far too complex to be listed only in grade 4. But that is exactly what is done in Core Standards. It is as though the authors had a master-list of topics and felt free to sprinkle them wherever there might have been room.

In grade 5 the analogue of 3-MD(5) is presented:

- 5-MD(3) Recognize volume as an attribute of solid figures and understand concepts of volume measurement.
- a. A cube with side length 1 unit, called a “unit cube,” is said to have “one cubic unit” of volume, and can be used to measure volume.
  - b. A solid figure which can be packed without gaps or overlaps using  $n$  unit cubes is said to have a volume of  $n$  cubic units.

Partly, I feel that this standard is occurring too early. It takes some time and effort for students to appreciate the complexity of visualizing solid figures through plane sections or possibly nets. Partly, as before, this standard is avoiding the real issues, namely, determining the volumes of figures that can not be decomposed into  $n$  cubes without gaps or overlaps, such as triangular prisms or rectangular cones. When we look at this pair of issues together, we can begin to see why I feel so uncomfortable with these standards.

At the same time, look at the geometry standards 5-G(3) and 5-G(4).

- 5-G(3) Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.
- 5-G(4) Classify two-dimensional figures in a hierarchy based on properties.

Except for parsing the convoluted language of the first, both of these standards are at an astoundingly trivial level for fifth grade. By this time students should be comfortable with the area formula for a triangle, and should be constructing compound two and three dimensional figures as well as determining a number of their properties.

In eighth grade the experimental approach to geometry that I mentioned earlier manifests for the first time. First there is a very superficial development of the properties of some Euclidian transformations in the plane:

- 8-G(1) Verify experimentally the properties of rotations, reflections, and translations:
- a. Lines are taken to lines, and line segments to line segments of the same length.
  - b. Angles are taken to angles of the same measure.
  - c. Parallel lines are taken to parallel lines.

Then, based entirely on the relatively weak standard above we are directly given one of the most subtle definitions of congruence we could possibly find.

8-G(2) Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

It is not the first piece of 8-G(2) that disturbs me – though there are a number of key steps that are hidden within it – but the second: given two congruent figures, describe a sequence that exhibits the congruence between them. (By “a sequence” I am presuming the writers meant “successively applying rotations, reflections and translations.”) What is being hidden here is the result that is deep even at the level of a university course in geometry: given two congruent figures, then there exists a Euclidean transformation that takes the first to the second, and a Euclidean transformation that takes the second to the first.

It is at the point above, and even more so with the corresponding similarity standard

8-G(4) Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two dimensional figures, describe a sequence that exhibits the similarity between them

where I feel that we are dealing with an experiment on a national scale. There are even more difficulties with the statement “given two similar two dimensional figures, describe a sequence that exhibits the similarity between them” than was the case with the corresponding statement in 8-G(2).

Before we dare to challenge teachers and students with standards like these, we absolutely have to test the approach in more limited environments, and I find it highly disturbing that H.-H. Wu, the main author of the geometry standards in Core Standards, feels able to make the following statement in a recent article:

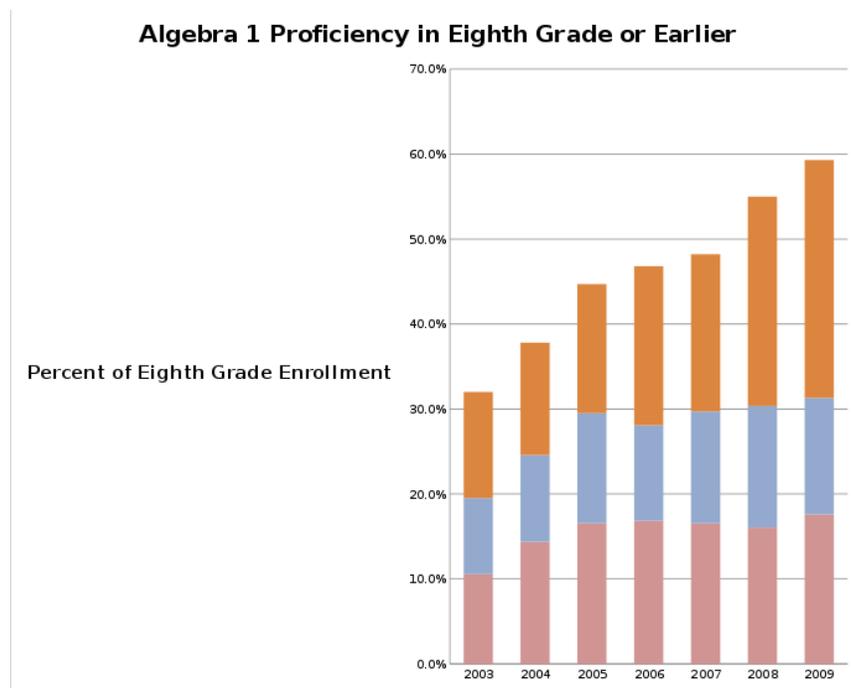
The mathematical coherence of CCMS also lies at the heart of the discussion of high school geometry. Briefly, the better standards, such as Californias, insist on teaching proofs. This is a good thing, but it does place an unreasonable burden on a high school course on geometry as the only place where any kind of proof can be found in school mathematics. As a result, some of these courses begin with formal proofs based on axioms from the beginning, with no motivation. There is another kind of reaction, however. Giving up entirely on proofs as unlearnable, some courses treat plane geometry as a sequence of hand-on activities that do not mention proofs. In addition, both kinds of courses are disconnected from the teaching of rigid motions (translations, rotations, and reflections) in middle school. What CCMS does is to add the teaching of dilations to rigid motions in grade 8 using hands-on activities, and on this foundation, develops high school geometry by proving all the traditional theorems. For the first time, the school geometry curriculum provides a framework in which all the apparently unrelated pieces of information now begin to form a coherent whole. It holds the promise that learning geometry in K–12 can finally become a reality.

Over the last 12 years Wu and I have collaborated on the California Framework, a number of other states standards, and on a number of nationally influential documents. Normally, Wu is very careful about distinguishing between what one *hopes is true* and what one knows will work, but in this instance I feel he has allowed his innate hope to overwhelm caution.

## Eighth Grade Algebra

Another issue with the Core Math Standards is that there are no provisions for eighth grade algebra. This contrasts with the California standards where the expectation is that most students will be ready for Algebra I by eighth grade.

Moreover, as the following graph shows, eighth grade Algebra I is basically working already, with almost 60% of California's students taking the course either in seventh or eighth grade.



It is worth noting that setting standards up so that Algebra I occurs naturally by eighth grade involves a large amount of preparatory material including basic pre-algebra standards and certain key geometry standards, such as understanding that the graph of a linear equation is a straight line. So it is far from sufficient to just list key algebra topics and decree a course that covers them.

As regards the Core Standards this is an issue I, as a member of the Pathways Committee, as well as the Validation Committee, have been struggling with for months. We have been able to rough in courses that are *mostly* based on the High School Core Standards, which will work, but we are far from finished with this project.

## Final Remarks

There had also been very real strengths in the document. Many of the discussions, among them ratio and rate in grade 6, and proportion in grade 7, had been excellent up to the May 25th draft. But there were dramatic changes there and in the final version. Perhaps the writing group responded sloppily to inaccurate critiques of the previous draft standards in this this area, but I do not really know. In any case there are now actual errors in the sixth and seventh grade discussions of ratios and rates. They had been clear and mathematically correct presentations of material that is typically very badly done in most state standards in this country. Now they are neither mathematically correct nor especially clear.

As far as I know the other strong areas through grade 8 in the May 25th draft remained strong in the final document, but I have not checked this assertion in detail.

In any case, in spite of all these issues, only the very best state mathematics standards, those of California, Massachusetts, Indiana and Minnesota are stronger than these standards. Most states would be far better off adopting the Core Math Standards than keeping their current standards. However, California, and the other states with top standards would almost certainly be better off keeping their current standards. But overall, I would not take this as an endorsement of the final version of Core Standards – rather I would regard it as a strong critique of the really poor quality of 90% of the state standards as of 2010.