FACT-CHECKING RESEARCH CLAIMS ABOUT MATH EDUCATION IN MANITOBA

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EXECUTIVE SUMMARY

In a Winnipeg Free Press article, *Mathematics education of Manitoba teachers should be based on research* (November 13, 2024), Dr. Martha Koch, an Associate Professor in the Faculty of Education at the University of Manitoba, made several claims about recent amendments to the Teaching Certificates and Qualifications Regulation under *The Education Administration Act*. These amendments significantly reduced the subject-area expertise required for teacher certification. Koch used the phrase "research shows" 15 times in her article. Some key claims put forth in the article include:

1. "The recent changes mean that Manitoba's teacher certification requirements are better aligned with current research in mathematics education."
2. "Notably, research shows that early and middle years teachers (grades K-8) who have taken more undergraduate university courses in mathematics are not more effective teachers of mathematics. That is, their students do not have better outcomes in mathematics."
3. "In fact, some studies have shown that K-8 students actually have lower achievement in mathematics if their teachers have more undergraduate courses in mathematics."

Since Koch's statements seemed dubious, she was asked to provide supporting evidence. She responded by circulating an eight-page research synopsis referencing 22 articles and books. After reviewing all 22 references, we found that none credibly support the above claims, and some even contradict them.

Additionally, Koch made statements about research on "mathematics knowledge for teaching" (MKT) in her Winnipeg Free Press article. The references she provided contain repeated, unambiguous statements emphasizing mathematical subject content knowledge as a necessary component of MKT—an important detail omitted by Koch.

The potential consequences of relying on claims that appear to lack evidence are significant, particularly given their possible influence on public policy affecting Manitoba children.

Our main findings

1. Faulty premises and conclusions not aligned with evidence

- Koch implied that pre-service K-8 teachers are being required to take standard undergraduate math courses—similar to those designed for physicists, mathematicians, and engineers—even though all Manitoba math departments offer specialized courses tailored for K-8 teachers.

- In several cases, Koch appears to draw conclusions that are not supported by the articles.

2. Lack of supporting evidence

- Not one article provided by Koch concludes that "K-8 students achieve lower outcomes when their teachers have more undergraduate math courses."
- Many of the articles appear to contradict Koch's claims, have been applied out of context, or are irrelevant to the discussion.
- Based on our analysis, the articles do not provide support for the idea that K-8 pre-service teachers should avoid math courses provided by university math departments.

3. Serious methodological issues

- Several studies clearly lacked proper design, or reported results that lacked statistical significance, making causal inferences impossible.

4. Disregard of contradictory evidence

- Several studies omitted by Koch indicate a positive correlation between math content courses taken by teachers and improved student achievement.
- Several of the articles Koch cited emphasize the need for stronger math content preparation for prospective teachers. One even referred to a recommendation for a minimum of six credit hours in math as an admission requirement for K-8 pre-service teachers—contradicting Koch's conclusions.

5. Impact of misinformation

- In the Manitoba Legislative Assembly on November 22, 2024, Tracy Schmidt, the Acting Minister of Education and Early Childhood Learning, stated that the amendments to *The Education Administration Act* "were based on research on math education, not on opinion." This raises concerns about the role of Koch's claims in shaping these policy changes, particularly given concerns about the lack of supporting evidence.

Conclusions and recommendations

Our detailed review discusses each of the cited papers, demonstrating that none appear to substantiate Koch's claims.

Given the serious implications of Koch's statements, and their potential impact on public policy, we make the following recommendations:

1) **Retraction:** Dr. Martha Koch should retract her Winnipeg Free Press article, as it gives readers the misleading impression that her claims are supported by research.

2) **Policy Caution:** The Manitoba government should consult more broadly, and exercise greater caution when relying on education research, to inform policy decisions.

ARTICLE ANALYSIS

Claims and context

Claims made in Martha Koch's Winnipeg Free Press article:

1. "Notably, research shows that early and middle years teachers (grades K-8) who have taken more undergraduate university courses in mathematics are not more effective teachers of mathematics. That is, their students do not have better outcomes in mathematics."
2. "In fact, some studies have shown that K-8 students actually have lower achievement in mathematics if their teachers have more undergraduate courses in mathematics."
3. "The changes that Manitoba Education has made are an important part of ensuring that mathematics education in Manitoba is reflective of the research."

As noted in the executive summary, Koch has incorrectly implied that pre-service K-8 teachers are being forced to take standard undergraduate math courses–like those taken by physicists, mathematicians and engineers. She then argued that requiring them to take standard undergraduate math courses (the faulty premise) is unhelpful or counterproductive and Koch appears to believe that the supplied articles support this assertion. In reality, however, pre-service K-8 teachers in Manitoba are not required to take standard undergraduate math courses. Instead, they are provided with specialized math content courses for elementary school teachers.

Our analysis addresses two key questions: a) Do the articles provided by Koch support her claims under the faulty premise? (Even though this premise does not reflect the actual policy.) b) Do the articles support her claims under the accurate premise—that pre-service K-8 teachers are provided with specialized math content courses?

Our analysis found that the references Koch provided do not credibly substantiate her claims under either premise. In fact, in many cases the references contradict her assertions.

The detailed analysis that follows examines each reference provided by Koch, offering both an analysis with respect to Koch's claims as well as a summary of each article.

Begle (1972, 1979)

Begle (1972)

Begle, E. G. (1972). *Teacher knowledge and student achievement in algebra.* School Mathematics Study Group (Report No. 9). Washington, DC: National Science Foundation.

Analysis: ***Does not support the claims.***

- This study does not consider the courses a teacher has taken in university in the first place, so is irrelevant to the claims.
- The study was done with Grade 9 students, while Koch's claims were about K-8 students.

- This study was an observational study and, beyond that, it has serious design flaws, as mentioned by the author.
- Design flaws aside, if the study were to be used to draw conclusions, the authors mention that "The small, but positive, correlation between teacher understanding of the real number system and student achievement in ninth grade algebra would lead to the recommendation that teachers should be provided with a solid understanding of the courses they are expected to teach." This is precisely the purpose of the math content courses we offer pre-service elementary teachers in Manitoba.

<u>Summary of study:</u> The author performs an experiment to determine to what extent a 9th grade math teacher's knowledge of algebra is correlated to the outcomes of their students. The teachers were given a test that measured both their knowledge of algebra on the real line as well as their understanding of abstract algebra (groups, rings, fields–topics covered in 3rd and 4th year math courses–not the type of courses Manitoba math departments offer pre-service K-8 teachers). Their students were given a test on 9th grade algebra.

Begle found that "*teacher understanding of modern algebra (groups, rings, and fields) has no significant correlation with student achievement in algebraic computation or in the understanding of ninth grade algebra. Teacher understanding of the algebra of the real number system has no significant correlation with student achievement in algebraic computation. However, teacher understanding of the algebra of real number system does have a significant positive correlation with student achievement in the understanding of ninth grade algebra. Nevertheless, while this correlation is statistically significant, it is so small as to be educationally insignificant*." There is no mention of "the number of undergraduate courses a teacher has taken" or what undergraduate courses they may have taken in the paper.

This study has serious experimental design flaws. The pool of teachers was not randomly chosen and teachers needed to apply to take part in the experiment. In Begle '79, the author applauds the teachers that chose to take part, noting that "*most teachers are reluctant, understandably, to submit to tests of their grasp of subject matter.*" Additionally, the teachers needed to apply for and gain acceptance to an NSF summer institute to be considered for the survey. It is highly unlikely that the teachers that took part in this survey are a fair representation of the overall pool of all 9th grade math teachers.

Further note: In our view, it is disingenuous to suggest that a study showing no correlation between teachers' knowledge about groups, rings and fields and their 9th-grade students' math performance justifies removing the requirement for pre-service K-8 teachers to take two specialized math content courses designed for elementary school teachers.

Begle (1979)

Begle, E.G. (1979). *Critical Variables in Mathematics Education.* Mathematical Association of America and National Council of Teachers of Mathematics.

<u>Analysis:</u> ***Does not provide credible support the claims. First, the study described in the book considered courses at the calculus level or higher, so this is an example of an attempt to provide evidence against the faulty premise mentioned in the Executive Summary. Apart from that, the***

study's observational design and lack of control for confounding variables undermine its validity, making it unsuitable for drawing causal inferences.

The study discussed in this book considers teachers who took credits at the calculus level or higher. Again, Manitoba math departments provided specialized courses for elementary -re-service teachers so it is inappropriate to attempt to apply this work to the Manitoba situation. It is unlikely that courses like UW MATH-2903/2904 and UM MATH-1080/1090 for elementary teachers even existed at that time.

Despite the fact that the study is irrelevant to the claims, there are other serious issues. In fact, it's hard to believe that this work, with its serious and obvious design flaws, is being cited as evidence of anything. Some of the flaws with this study are as follows:

- It was an observational study and not an experimental study.
- Students and teachers were not randomly assigned to specific conditions or interventions.
- There was no control for confounding variables, of which there are many (e.g. differences in teaching methods, students' home environment, socioeconomic status, school resources, classroom dynamics, etc.)
- Without random assignments, it's impossible to rule out other factors that could be influencing both teacher characteristics and student achievement.
- The study relied on undergraduate course counts (credits at the calculus level or higher). The type of courses taken, grades received in those courses and other important variables appear not to have been considered at all.
- The data for the study was collected during 1962–1967.
- It's incomprehensible to us that a study that considered teacher characteristics in 1962 is being cited to support claims about teacher characteristics in 2024. For example, we suspect the average teacher candidate at that time likely entered university years earlier with solid knowledge of K-8 math. This isn't the case with a lot of high school graduates in Manitoba.

A review written by Geoffrey Howson from 1980 echoes our concerns about experimental design.

"*Even Begle's other major source, the enormous NLSMA investigation, can be called into question on some counts. It was, according to Begle, a 'naturalistic' study; more explicitly it was 'an observational study, not an experiment; hence there was no randomization of students or textbook groups' [4]. As a result, "disproportionate numbers of schools were selected from certain geographical areas (e.g. California) and from among SMSG users'.*" (p. 61)

See Howson, Geoffrey. *The Mathematical Gazette* 64, no. 427 (1980): 64–68. https://doi.org/10.2307/3615899.

For the interested reader, SMSG refers to the School Mathematics Study Group, often referred to as the "new math" project of the 1950s-1970s. The project itself was a failure and was completely abandoned by the mid-1970s. Begle was director of Yale's SMSG.

Summary of book: This 1979 book attempts to provide a rather complete survey of empirically backed surveys within the field of mathematics education at the time of publication. Begle had a poor opinion of non-empirical research in the field, writing: "*much of what goes on today in mathematics education is based on opinions that are so firmly held that the thought of doubting crosses few minds. Yet most of* *these opinions have no empirical substantiation, and in fact many of them are, if not wrong, at least in need of serious qualifications.*" From this book, only Chapter 3: 'Teachers' is relevant to any claims made by Koch.

In Chapter 3, Begle notes that predicting the effectiveness of a math teacher (K-12) is a difficult problem. For example, Begle mentions that the scores a teacher receives on typical evaluations performed by principals/administrators have little or no relation to the mathematical improvement in their students. Begle discusses data collected from the NLSMA reports (National Longitudinal Study of Mathematical Abilities) on teacher effectiveness in mathematics. These NLSMA reports analyzed the relationship between teacher characteristics and student achievements. Teacher characteristics included number of years of teaching, highest academic degree, current marital status, major field, etc. In terms of math background, the relevant characteristics were credits in math methods courses, math credits **beginning with calculus**, and math as a major or minor.

As an interesting side note, something that was not mentioned by Koch is that the study found that *none of the teacher characteristics considered in the report were found to be significant indicators of teacher effectiveness* – including math methods (pedagogy) courses!

Shulman (1986)

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Analysis: ***Does not support the claims.*** No empirical evidence is discussed in this paper. Furthermore, nowhere in this article does Shulman argue against pre-service teachers taking math content courses.

Summary: The author begins by describing how the requirements for becoming a teacher in the USA during the 1870s were primarily based on content knowledge with little focus on pedagogy whereas the requirements for teacher certification in the 1980s (the time at which the article was written) were primarily based on pedagogical knowledge with little emphasis on subject knowledge. The author then describes how pedagogy and subject knowledge were intertwined during the Middle Ages. At several points throughout the paper, the author argues that teachers require both pedagogical and subject knowledge and, furthermore, that teachers need to understand the "why". The author also points out that a flaw of current educational research (at the time of writing) was that it tended to be too focused on pedagogical methods and placed too little emphasis on subject knowledge of the teacher.

The author argues in the latter part of the paper that "case knowledge", which is "*specific, well-documented, and richly described events*" (typically stories about a teacher doing something unexpected in their class that has an educational benefit) should make up part of the education literature as well, and that these can be used to inform future decisions of teachers. Near the end of the paper, the author expresses that they are in favour of examinations for becoming a teacher: "*I firmly believe that we must develop professional examinations for teachers... They must reflect an understanding that both content and process are needed by teaching professionals*". The author argues that the particulars of the exam should be "*informed by a well-organized and codified case literature*". The author discusses 'pedagogical content knowledge,' (PCK) which is knowledge of content specifically for teaching. This includes knowing "*ways of representing and formulating the subject that make it comprehensible to others*" and "*understanding of what makes the learning of specific topics easy or difficult.*"

This is not an argument against taking math content courses; in fact, one would realistically assume that strong content knowledge should precede strong pedagogical content knowledge.

Ma (1999; 2010)

Ma, L. (1999; 2010) *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Routledge.

Analysis: ***Does not support the claims***. In fact, Koch's recommendations seem to **contradict** some of Ma's work. In addition, *Koch neglects to acknowledge key aspects of Ma's findings and recommendations*. While Koch has argued in the *Winnipeg Free Press* that pedagogy courses alone are sufficient, Ma insists that K-8 teachers must also have a "profound understanding of fundamental mathematics."

Ironically, Ma's book helped to inform the design of our math content courses for K-8 pre-service teachers at the University of Winnipeg. It is this profound understanding of fundamental mathematics that we strive for in our courses. Since Koch gives the impression that she holds Ma's work in high regard, it's confusing that Koch doesn't acknowledge that Manitoba math department's content courses for elementary teachers are aligned with Ma's work.

In her research synopsis, Koch notes that "Ma studied the mathematical understanding of elementary teachers in China" who had the equivalent of Grade 9 education and two or three years of teacher preparation in comparison to US elementary teachers who had Grade 12 diplomas with one or more university degrees. Ma found that Chinese teachers displayed a more comprehensive knowledge of the math taught in elementary school.

It would appear that Koch thinks this comparison between Chinese and US teachers supports her claim that math content courses for pre-service K-8 teachers are unhelpful. However, *Koch is again misrepresenting the type of course for K-8 teachers we offer in Manitoba math departments and Koch neglects key differences between North American and Chinese math education*.

- Students in China (note that this was in 1999) have a more solid grasp of elementary mathematics than US students in the first place. This has been reflected repeatedly in international tests such as TIMSS. Our courses are designed to address the gaps that many Manitoba students in pre-service K-8 education programs have when it comes to elementary education.
- Based on Ma's work, it seems more reasonable to reach the opposite conclusion of Koch's—math content courses that focus on providing a thorough understanding of the math taught in elementary school and a solid grasp of those fundamental concepts are valuable.
- It is unlikely that students in the US comparison group had taken the type of course we offer in math departments at Manitoba universities for pre-service K-8 teachers; if they did it is not mentioned in Ma's work.

Further summary of Ma's work and notes on how our courses align with Ma's work:

- In Chapters 2 & 3 of Ma's book, Ma discusses her studies on Chinese teachers and US teachers' interaction with standard algorithms for subtraction and multi-digit multiplication and notes that US teachers tended to view these more procedurally. Ma states that "*The Chinese teachers were concerned with knowing why algorithms make sense as well as knowing how to carry them out.* ***Their attitudes were similar to those of practicing mathematicians***." (Pg. 124)

  First note that Ma acknowledges that mathematicians are concerned with both how and why algorithms work. As mathematicians, we can confirm this is true. As well, UW MATH-2903/2904 and UM MATH-1080/1090 teach conceptual understanding of the standard algorithms to pre-service K-8 teachers, which Ma notes is important.
- In chapter 4, Ma discusses her study on Chinese teachers and US teachers' interaction with fraction arithmetic. She notes that US teachers were often unable to represent or explain the rationale behind fraction operations. Further she states, "***this suggests that in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it***."

  In other words, knowing how to teach something is not sufficient. Ideally, one should understand the underlying structure. Understanding of operations with rational numbers is included in our courses for pre-service K-8 teachers.

Articles authored by Ball et al.

Summary of articles and understanding MKT

<u>**Overall Analysis:**</u> ***It appears that Koch has either misunderstood some of the findings in this research or has drawn conclusions that are not supported by the articles. From her writing, we get the impression that Koch has misunderstood the definition of "math knowledge for teaching" (MKT).***

Having read the Ball papers, we can definitively say the following:

- These papers do not support Koch's claim that prospective K-8 teachers should not take math courses from a math department.
- These papers support prospective K-8 teachers taking courses like UW MATH-2903/04 and UM MATH 1080/90, which, using the terminology of the Ball papers, were explicitly designed to ensure that students obtain *both* the common and specialized MKT that the Ball papers suggest are needed for effective K-8 math teaching.

In her research synopsis, Koch writes "Ball's extended research agenda informs my view of research-based B.Ed. admissions requirements for Manitoba." It is thus our impression that she may view the Ball papers as offering strong arguments in support of her position. However, upon reading these papers, it appears that Koch has either misunderstood this research or has drawn conclusions that are not supported by the articles.

At the heart of the series of Ball (and other) papers is the notion of "Mathematical Knowledge for Teaching (MKT)." Primarily, these papers study the impact of certain aspects of a teacher's MKT – common and specialized MKT – on how effective they are at teaching elementary school (K-6) children math. Since the notion of MKT is central to the series of Ball papers, and Koch seems to have misunderstood the definition of MKT, it is essential that we carefully examine this series and consider the authors' intended meaning of MKT.

The most concise definition of mathematical knowledge for teaching (MKT) used by the authors seems to be found in Ball, Hill & Bass 2005, p. 22:

> *"We defined mathematical content knowledge as being composed of two key elements, "common" knowledge of mathematics that any well-educated adult should have* ***and*** *mathematical knowledge that is "specialized" to the work of teaching and that only teachers need to know."*

Note that the authors emphasized the word "and" in the above quote; the definition of MKT includes *both* basic math knowledge/skills, and additional knowledge that a teacher should know to help them teach math to children. On page 377 of Hill, Rowan & Ball 2005, the authors explain their use of mathematical knowledge used in teaching elementary school mathematics (MKT) this way [here the emphasis is ours to help highlight what Koch seems to have dismissed]:

> *"With the phrase "used in teaching," the developers of this instrument meant to capture* ***not only*** *the actual mathematical content that teachers teach (e.g., decimals, area measurement, and long division) but also the specialized knowledge of mathematics needed for the work of teaching.* ***"Specialized" content knowledge, as we define it, is mathematical knowledge, not pedagogy****. It includes knowing how to represent quantities such as 1/4 or .65 using diagrams, how to provide a mathematically careful explanation of divisibility rules, or how to appraise the mathematical validity of alternative solution methods for a problem such as 35 x 25."*

So, common MKT is the math knowledge and skills teachers must know, as asserted in the Ball papers, concerning how to solve mathematical problems by using standard mathematical tools, knowing basic math facts, knowing how to perform fraction arithmetic, knowing how to compute ratios and percentages, and so on. Specialized MKT involves having a deeper understanding about why a procedure or algorithm works with, say, integers or fractions, and knowing different ways to represent a mathematical operation (e.g., symbolically and visually) to aid a teacher's ability to help students learn. **As the authors explicitly wrote in the above quote, both common and specialized MKT are a type of *mathematical knowledge–not pedagogy.***

Here are some additional illuminating quotes regarding the intended meaning of common and specialized MKT from Ball, Hill & Bass 2005: Regarding skill/knowledge (common MKT), on p.17 they write that it was

> *"obvious that teachers had to know the topics and procedures they teach—factoring, primes, equivalent fractions, functions, translations and rotations, and so on"* and *"clearly, being able to multiply correctly is essential knowledge for teaching multiplication to students."*

Regarding specialized MKT, following their discussion about how to best teach the standard algorithm for multiplication, they wrote on p. 21 that

> "*each of these common tasks of teaching involves mathematical reasoning as much as it does pedagogical thinking.*"

The courses UW MATH-2903/04 and UM MATH-1080/90 were explicitly designed to ensure prospective K-8 teachers have both common and specialized MKT, according to the definitions of these concepts from the Ball papers. The interested reader can check that this is so by reading the preface of the textbook used in all of these classes, *Mathematics for Elementary School Teachers* by Ricardo D., Cengage, ISBN: 9781133983606. We have included a copy of the preface in Appendix B.

Quoting from the first of the Ball papers, on p. 3 of Koch's research synopsis, she wrote the following about MKT:

> *"MKT is "a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations, such as engineering, physics, accounting, or carpentry"* (Ball, Hill & Bass, 2005, p. 17).

Moreover,

> *"knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably"* (p.22).

Based on her writing, it is our impression that Koch has somehow interpreted this to mean that MKT can only be developed in courses offered by Faculties of Education, which is not a conclusion drawn in the Ball articles, nor is it implied by the Ball articles.

It also seems that Koch may be unaware that a mathematician's profession involves knowing, understanding and explaining math. Indeed, as those who make new mathematical discoveries, and teach math, we are exclusively in the business of understanding and explaining mathematical concepts, and mathematical understanding is a *major focus* of our classes. This is particularly true of our classes designed for prospective K-8 teachers, UW MATH-2903/04 and UM MATH-1080/90, which focus on providing a deep understanding of the K-8 math curriculum.

We now dive more deeply into the contents of each of the individual Ball papers referenced by Koch.

Ball, Hill & Bass (2005)

Ball, D.L., Hill, H.C. & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 30*(3), 14-17, 20-22, 43-46.

Analysis: ***It appears that Koch has either misunderstood parts of this paper or has drawn conclusions that are not supported by the article. Based on her writing, it is our impression that Koch has misunderstood the definition of "math knowledge for teaching" (MKT).***

Summary of the authors' main questions, methods, and main conclusions:
The first 5 pages of this ten-page paper are meant to clarify what the authors mean by "common and specialized Mathematical Knowledge for Teachers MKT" (summarized above). An explicit description of the questions the authors attempted to answer in this paper, their research methods, and conclusions begin on p. 22.

The two main questions (p. 22) were:

- Is there a body of "mathematical knowledge for teaching" (MKT) that is specialized for the work that teachers do?
- If so, does MKT have a demonstrable effect on student teaching?

The goal seemed to be to answer these questions for K-6 teachers (not K-8), but it seems that only grades 1 and 3 teaches were studied, (so grades 4-8 teachers were not studied).

The authors created 250 multiple choice questions. As noted above, the authors considered two key elements of MKT: "common MKT" and "specialized MKT." They divided their questions into two groups based on their opinions about what constituted knowledge from each category. Their questions focussed on the numbers and operations strand of the elementary school curriculum, and they state that they attempted to write "ideologically neutral questions." It's hard to evaluate the quality of the questions since they do not provide a complete list, but they do give one example of a common MKT question and one example of a specialized MKT question. (The example given for a common MKT question is a bit odd, since it tests non-essential mathematical knowledge for a numerate citizen, but it is clear that UW MATH-2903/04 and UM MATH-1080/90, or most other university-level math courses, would help students answer both questions correctly.)

They gave their MKT test of 250 questions to 700 first- and third-grade teachers, and they collected the scores of their students (nearly 3000 of them) on the math portion of the "Terra Nova" standardized test. (The authors claim that the Terra Nova is reliable and valid, but don't seem to provide evidence of this.) The authors then determined whether higher scores for teachers on their 250-question MKT test predicted better student math scores on the Terra Nova.

Though they do not provide any raw data or statistical figures for the reader to examine, they state that first- and third-grade *"teachers' performance on our knowledge for teaching questions – including **both** (our emphasis) common and specialized content knowledge – significantly predicted the size of student gain scores"* (p. 44).

Some additional comments about this paper

The authors conclude that both common and specialized MKT are important for first-and third-grade teachers. Courses such as UW MATH-2903/04 and UM MATH-1080/90 designed for Early and Middle Years Teachers cover *both* common and specialized MKT, according to the definitions in this paper (see above). Indeed, here are some additional quotes from and commentary on this paper.

From page 16: [The emphasis, which is ours, is meant to highlight what Koch has not emphasized.]

"*Although many studies demonstrate that teacher's mathematical knowledge helps support additional achievement, the actual nature of that knowledge - whether it is simply basic skills at the grades they teach or complex and professionally specific mathematical knowledge - is largely unknown. The benefits to student learning of teachers'* ***additional*** *coursework either in mathematics itself or "mathematical methods" — courses that advise ways to teach mathematics to students—are disputed by leading authorities in the field*."

In conclusion,

- The authors stated that many studies demonstrate that content knowledge helps support student achievement. They do not argue against any type of math-content knowledge anywhere in this paper.
- The authors are aiming to determine how much *additional* content knowledge a teacher needs beyond the level at which they're teaching.
- The classes UW MATH-2903/04 and UM MATH-1080/90 focus primarily on content knowledge associated with the K-8 curriculum.

From page 17 [The emphasis, which is ours, is meant to highlight what Koch has ignored.]

"While it seemed obvious that teachers had to know the topics and procedures they teach—factoring, primes, equivalent fractions, functions, translations and rotations, and so on*—our experiences and observations kept highlighting* ***additional*** *dimensions of the knowledge useful in classrooms. In keeping with this observation,* ***we decided to focus our efforts on bringing the nature of this additional knowledge*** *to light, asking what, in practice, teachers need to know about mathematics to be successful with students in classrooms."*

In conclusion,

- They have explicitly stated that it's *obvious* that the first thing a teacher needs is content knowledge of what they're teaching. They need to "know the topics and procedures they teach." We agree that this should be obvious to everyone.
- As we know firsthand, many Manitoba high school graduates (particularly those who have not taken Precalculus Math 40S) cannot be assumed to be fluent with the procedures that a K-8 teacher will teach. It cannot be assumed that they are fluent with factoring, fraction arithmetic, et cetera. That is, many high school graduates lack even "common MKT".
- Notice the words "additional knowledge" (used repeatedly in the paper). The authors are focussed on what K-8 teachers need *in addition* to the "obvious" fact that they need to know the topics and procedures that they teach.

On pages 17 – 20, the authors aim to further explain their ideas about "common MKT" and "specialized MKT" using their ideas about the effective teaching of the standard multiplication algorithm to children. (It is clear to us that the authors believe that the standard algorithms should be taught, and taught effectively.) Here is another quote from the paper:

> "*Clearly, being able to multiply correctly is essential knowledge for teaching multiplication to students. But this is also insufficient for teaching. Teachers do not merely do problems while*

students watch. They must explain, listen, and examine students' work. They must choose useful models or examples. Doing these things requires additional mathematical insight and understanding. "

In conclusion,

- The authors are referring to K-8 math content knowledge and skills as "*essential knowledge*". (We agree, and this is why K-8 math content knowledge and skills are taught in UW MATH-2903/04 and UM MATH-1080/90.)
- They say the above is insufficient. However, it is still "*essential knowledge."*
- Understanding of K-8 math content is also needed for effective teaching. We agree, and teaching this understanding so that teachers are in a position to be able to "explain, listen, and examine students' work" and "choose useful models or examples" is *the main purpose* of UW MATH-2903/04 and UM MATH-1080/90.
- The authors also argue for some instruction on how to best teach the algorithm (pedagogy). The steps they show on page 20 would essentially be the way that we would show the students how the algorithm works in UW MATH-2903/04 and UM MATH-1080/90.
- The example on pages 17–20 supports an argument for K-8 teachers taking courses like UW MATH-2903/04 and UM MATH-1080/90.

<u>Our conclusion</u>

This article does not support Koch's claims that students should not be required to take university-level math courses from a math department. It seems clear to us that the authors believe that precisely the sort of math content knowledge for teachers covered in the classes UW MATH-2903/04 and UM MATH-1080/90 is important for prospective elementary school teachers.

Hill, Rowan & Ball (2005)

Hill, H.C., Rowan, B. & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

<u>Analysis:</u> ***Does not support any of the claims made by Koch.***

It is unclear why Koch seems to think this article supports her arguments.

- The authors stated that the number of math courses taken by grade three teachers positively, though not quite significantly, predicted student gains in the third grade. However, the study pooled together both math content courses and math pedagogy courses so no causal inference can possibly be made about whether either contributes to gains in students' math achievement.
- Apart from the above, there were other issues with the study design, some of which the authors note themselves.
- Apart from issues with experimental design, teachers' math knowledge for teaching, as determined by their scores on a CKT-M questionnaire, positively predicted mathematical achievement gains of first- and third-grade students on the Terra Nova test. As already

discussed, our courses UW MATH-2903/04 and UM MATH-1080/90 are designed to provide MKT so, even if we were to ignore the limitations of the study, this finding supports requiring prospective elementary teachers to complete UW MATH-2903/04 or UM MATH-1080/90.

In this paper, the authors used the abbreviation CKT-M for content knowledge for teaching mathematics, rather than MKT.

Summary of the authors' main questions, methods, main conclusions, and reservations

The purpose of this work was to explore whether and how teachers' MKT contributes to gains in students' mathematics achievement. The study included 115 elementary schools, 334 first grade teachers and 365 third grade teachers. Eight students were randomly selected from each class for a total of 1190 first graders and 1773 third graders. Data was compiled over a two-year period. Data on students, and their parents' SES and education levels, were obtained via assessments and parent interviews. Teacher information was obtained from structured self-reported teacher logs that included information about the teacher's experience, completed math content and math methods coursework, teacher certification status, and class time devoted to math instruction.

A teachers' content knowledge was measured by their score on 30 multiple choice questions, which (some) teachers completed as part of their teacher information logs. The multiple-choice question topics were balanced across content domains (13 number items, 13 operations items, and 4 pre-algebra items) and specialized (16 items) and common (14 items) content knowledge. The authors reported that teachers typically answered fewer than 30 items.

Teacher effectiveness was measured by calculating student gains made on two sittings, approximately one year apart, on the math portion of the Terra Nova standardized test.

The data about a teacher's prior math coursework was not carefully collected in this study. As part of the teacher logs, teachers were asked to self-report on the total number of math content and math methods courses they had taken but were not asked to distinguish between the number of courses of each type, and the authors did not attempt to determine how well the teachers did in these classes. The authors note that math-content courses taken by elementary school teacher candidates are typically taught by a member of a math department and cover mathematical topics found in the K-6 curriculum (like UW MATH-2903/04 and UM MATH-1080/90); math methods classes are usually offered by faculties of education and tend to focus on topics of pedagogical concern in an elementary school math class.

The authors concluded that teachers' mathematics knowledge for teaching, as determined by scores on their CKT-M (=MKT) questionnaire, positively predicted mathematical achievement gains of first- and third-grade students on the Terra Nova test. Grade one students of teachers with CKT-M scores in the lowest two deciles, and grade three students of teachers with CKT-M scores in the lowest three deciles, had significantly worse Terra Nova score gains. That is, students who had the least knowledgeable teachers appeared to learn less math than the students who had more knowledgeable teachers.

The authors found that teachers' mathematics preparation, based on the total number of teacher-reported completed math (content plus methods) courses positively (though this was not quite statistically significant at the 5% level) predicted student gains in the third grade.

The authors attempted to not overstate the strength of the conclusions that can be drawn from their study. On p. 390, they wrote *"In summary, this study involved a number of data issues, including the small number of students with complete data within each classroom, missing data on many variables, lack of complete alignment between the teacher and student mathematics assessments, and student attrition."* This statement was repeated in the first line of their conclusion on p. 399, where they also wrote *"we are less confident in any borderline or null results, such as those found for the teacher preparation measures."*

Some comments

It's not clear why the authors didn't attempt to obtain better data on the math methods and content courses taken by the teachers in the study, such as the exact number of methods versus content courses, and the grades the teachers obtained in these classes. The authors did not try to determine the relationship between a teachers' performance in these classes, their CKT-M scores, and student achievement. ***These authors did not attempt to disparage or devalue either math content or math methods classes in any way***. Beyond noting that their data seems to suggest that these courses (without distinguishing between content and methods classes) positively predict student achievement (again, this was not statistically significant at the 5% level), the paper does not do an effective job of testing their impact.

Beyond the many problems with the study design and data that were identified by the authors, it should also be noted that to account for "substantial amounts of student attrition and missing data on key variables," they used mean imputation methods, which could potentially lead to biased estimates. Mean imputation is a very simple method, but its simplicity comes at the cost of introducing bias. There are more sophisticated techniques such as multiple imputation, regression techniques, maximum likelihood estimation, or Bayesian methods which are generally preferred when dealing with missing data because they provide more accurate and unbiased estimates.

The authors did not attempt to determine anything about instructional methods (e.g., direct versus inquiry) used by the teachers in this study, and whether there was a relationship between the methods used, MKT and student achievement.

The authors make some odd comments about common versus specialized MKT on pages 387–8. They note again that both common and specialized MKT are entirely mathematical, not pedagogical, (based on their definitions of common and specialized MKT, we agree) but state the opinion that specialized MKT will be less familiar to those who do not teach. When it concerns math students, and certainly mathematicians, we disagree with this statement, which is simply an opinion put forward by the authors: any marginally good university-level math student would have no trouble with the specialized math example they give in the article. Math students are trained to approach all math with skepticism and to constantly evaluate their work, and their instructors' work, for errors, concision, elegance and generalizability.

Only grades one and three teachers and their students were studied. The authors do not study the importance of MKT in grades 4-8 teachers, which is relevant to the Manitoba situation. The authors noted that more mathematical knowledge is needed to teach grade three than grade one. We suspect that the authors would also agree that much more knowledge is increasingly needed to teach grades four, five, six, seven and eight math.

Our conclusion

This article does not support Martha Koch's claims that students should not be required to take university-level math courses from a math department. The sort of math content knowledge for teachers covered in the classes UW MATH-2903/04 and UM MATH-1080/90 is important for prospective elementary school teachers and all topics covered in these courses fall well within the scope of common and specialized MKT for K-8 teachers.

Ball, Thames & Phelps (2008)

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407.

Analysis: ***Not an empirical study. No causal inferences can be drawn from this study.***

This article is partly a survey paper, partly a theoretical exposition, partly an opinion piece and partly a speculative musing. It does not report on any one study and no data is included; it therefore cannot, and does not, support any of Koch's claims.

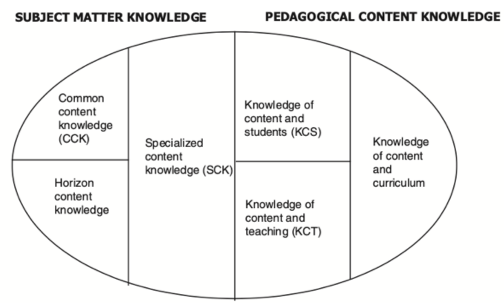
The paper begins with a history and description of Shulman's theory of pedagogical content knowledge (PKT). The remainder of the paper is primarily a discussion of how mathematical content knowledge (MKT) integrates within, and (especially) is different from, the general theory of PKT. As described earlier in this document, two subcategories of MKT, common and specialized, were studied in the 2005 Ball (and other) papers. Here, the authors again argue that *both* common *and* specialized MKT are essential types of knowledge for effective math teaching.

The authors spend the most time discussing specialized MKT and offer the opinion that it is a type of mathematical knowledge that is needed exclusively in math teaching. They stress that according to their definitions, both common and specialized MKT are types of mathematical knowledge and that this knowledge is independent of pedagogy and knowledge about students; as such they argue that specialized MKT is different from the PKT studied by Shulman and others.

Having identified MKT exclusively as a type of math subject knowledge, they add a new subcategory of math subject knowledge for teaching, "horizon content knowledge", which they argue is also essential knowledge for a teacher: horizon content knowledge is knowing what your students will learn in future grades. For example, they argue that a grade one teacher needs to know what is to be learned in grades two and three math classes, so that the teacher is aware of the most important topics to stress when teaching the first grade. (Similarly, a teacher of sixth-grade math needs to know the math from grades 7 and 8; a teacher of eighth-grade math needs to know the math content from grades 9 and 10, and so on.) We agree with their opinion that this is important content knowledge for teachers to have.

On p. 40, the authors include a diagram showing all of the subdomains of mathematics content knowledge for teachers (MKT), as theorized by Shulman, and added to in the series of Ball papers (see below). Note that what was called common and specialized MKT in previous papers are called common content knowledge (CCK) and specialized content knowledge (SCK) in this paper. They say that it would be interesting to study which of these subcategories of teacher knowledge are most essential for effective math teaching. All discussions are in the context of elementary school teaching; most examples and references come from grades K-3.

Figure 5
Domains of Mathematical Knowledge for Teaching



The debate about teacher-certification requirements in Manitoba is independent of the theorized subdomains of MKT sitting beneath the heading "Pedagogical Content Knowledge" found on the right side of the above figure.

The authors use significant space discussing why they think that specialized MKT is a skill exclusively possessed by teachers. This discussion falls into the category of opinion.

It is our opinion that the claims made, which are the authors' opinions, were often over-stated and self-flattering. An example of this is found on p. 397, where the authors discuss the (obvious) importance of a teacher being able to not only identify student errors but also determine what misunderstandings led to the errors, and that teachers must be able to identify these errors and misunderstandings quickly. They write:

> *"Although mathematicians engage in analyses of error, often of failed proofs, the analysis used to uncover a student error appears to be related to, but not the same as, other error analyses in the discipline. Furthermore, whereas teachers must process such analyses fluently, no demand exists for mathematicians to conduct their work quickly."*

Our opinion, as mathematicians, is that this is an absurd statement. Mathematicians do need to work quickly – efficiency is always a goal – and mathematicians not only look to find errors constantly and everywhere, but they also always aim to determine what misunderstandings led to errors when they have been made. Comparing the difficulty level involved in error-checking a complex mathematical proof with error-checking grade school math problems is, in our opinion, preposterous. Any mathematician, and indeed any *marginally* competent math major, would have no trouble with this, or any of the other task-examples of specialized MKT contained in this article. For another example, see the question in figure 4 from page 400. It is our opinion that specialized MKT is not quite as special as these authors might think it is.

Mostly, it seems that this paper is composed of (dressed-up) common-sense arguments about the importance of mathematics content knowledge for teachers. The contents of UW MATH-2903/04 and UM MATH-1080/90 is entirely contained within the types of common, specialized and horizon MKT that teachers need, according to the definitions found in this, and the other, Ball papers to which we were referred.

Below, we include a small sample of the many quotes from the paper that clearly show that the contents of UW MATH-2903/04 and UM MATH-1080/90 would support the development of common and specialized MKT, as these concepts are defined by these authors.

> From p. 395 [our emphasis]: *"Because it seemed **obvious that teachers need to know the topics and procedures that they teach—primes, equivalent fractions, functions, translations and rotations, factoring, and so on**—we decided to focus on how teachers need to know that content."*
>
> From p. 396 [our emphasis; necessary means essential, not sufficient means more is needed]: "*We start with this pure computational task [of using the standard algorithm for subtraction] because **teachers who teach subtraction must be able to perform this calculation themselves**. ... However, being able to carry out this procedure is necessary, but not sufficient, for teaching it*."
>
> From p. 399 [our emphasis]: *"By "common [MKT]," however, we do not mean to suggest that everyone has this knowledge. Rather, we mean to indicate that this is knowledge of a kind used in a wide variety of settings—in other words, not unique to teaching. **When we analyzed videos of teaching, it was obvious that such knowledge is essential. When a teacher mispronounced terms, made calculation errors, or got stuck trying to solve a problem at the board, instruction suffered and valuable time was lost**. In mapping out the mathematical knowledge needed by teachers, we found that an **understanding of the mathematics in the student curriculum plays a critical role in planning and carrying out instruction."***
>
> From p. 402 [Our emphasis. A correct answer to all of these questions is: in the courses UW MATH-2903/04 and UM MATH-1080/90.]
>
> *"Where, for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Where do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime or how and why the long division algorithm works? **Teachers must know these sorts of things and engage in these mathematical practices themselves when teaching. Explicit knowledge and skills in these areas are vital for teaching."***
>
> From p. 404, first paragraph of the conclusion: *"Teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher competency. The reason is simple: **Teachers who do not themselves know a subject well are not likely to have the knowledge they need to help students learn this content**."*

Conclusions

This article does not report on any one study and there is no data included; it is part survey paper, part theorizing, part opinion. It therefore does not, and cannot, be used in support of Koch's claims.

The contents of UW MATH-2903/04 and UM MATH-1080/90 is entirely contained within the domains of common, specialized and horizon MKT that teachers need, according to the definitions found in this, and the other, Ball papers. For evidence of this, the reader is referred to the preface of the textbook used in these classes, *Mathematics for Elementary School Teachers,* by R.D. Fierro (posted in the Appendix). As such, this article supports the idea of prospective K-8 teachers taking courses like UW MATH-2903/04 and UM MATH-1080/90.

Hill, Ball & Schilling (2008)

Hill, H., Ball, D. L., & Schilling, S. G. (2008). Unpacking "pedagogical content knowledge": Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372–400.

Analysis: ***Article is irrelevant to the claims.***

On p. 377 of this article, the authors again include the following diagram showing a "domain map" for mathematics content knowledge for teachers (MKT):

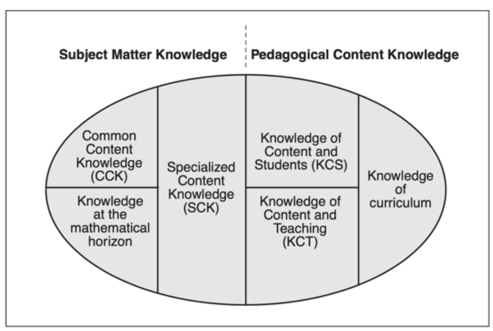


Figure 1. Domain map for mathematical knowledge for teaching.

These subdivisions are based on both Shulman's theory of pedagogical content knowledge (PCK) and Ball's (and others') papers on common and specialized MKT (abbreviated by CCK and SCK, respectively, in Figure 1). As noted earlier in this document, Ball and others strongly believe that *both* common and specialized MKT are essential components to effective teaching of mathematics. Based on the definitions of common and specialized MKT, (which seem to have been misunderstood by Koch), we agree. This paper is not about either common or specialized MKT. The debate about teacher-certification requirements in Manitoba is independent of the theorized subdomains of MKT sitting beneath the heading "Pedagogical Content Knowledge" found on the right-hand side of the above figure. In particular, the debate in Manitoba is independent of the theorized existence of Knowledge of Content and Students (KCS) in math – content knowledge intertwined with knowledge about how students think about, know or learn math – as a unique subdomain of MKT, which is the topic of this paper. So, it is not clear why Koch included this in her list of references. This paper is not relevant to this debate.

Here, nonetheless, is a brief synopsis of this paper.

Summary: This paper primarily reports on a first attempt to find methods that uniquely identify the level of a teacher's KCS, (in isolation of the other types of MKT found in the figure). The authors hoped to develop methods that could be practically implemented in a future large-scale study (not this study). Thus motivated, the authors aimed to develop a multiple-choice questionnaire to measure KCS, since such tests are easy to administer and grade. They believed that an effective measurement tool, if developed, would be of theoretical significance to math educators since it would help isolate KCS as a distinct subdomain of PCT/MKT. They also argued that one could use their tool, if effective, to see whether stronger KCS predicts effective math teaching. (Both motivators strike us as being more of esoteric importance to math educators' interests in Shulman's theoretical notion of KCS rather than the actual development of effective math teachers.)

The goal of this paper was not to determine whether KCS actually improves teacher effectiveness, just whether or not it could be detected in teachers. The authors found that their multiple-choice questionnaire approach had only limited success in predicting KCS in teachers: when providing correct answers, some teachers used KCS but others used mathematical reasoning (CCK) or simple test-taking skills.

Conclusions

This paper is not relevant to the current debate. While we obviously agree that teachers should have knowledge about how their students think about, know, and learn math, this article certainly does not, in any way, support any of Koch's claims.

Charalambos et al. (2011) & the cautionary tale of Vonda

Charalambos, C. Y., Hill, H. C., & Ball, D. L. (2011). Prospective teachers' learning to provide instructional explanations: How does it look and what might it take? *Journal of Mathematics Teacher Education*, *14*(6), 441–463.

Analysis: ***Study design does not allow for causal inferences. If anything were to be concluded from this case study, however, in our view it would be that math content knowledge is a necessary prerequisite for effective pedagogy. Indeed, consider the case of Vonda.***

This article presents a case study of four students enrolled in two courses offered by the first-named author as part of a one-year MA degree in education at a large midwestern American university. The authors had complete data from 16 students, from which they made a "purposeful selection" of four students to discuss in this article: Nathan, Kimberly, Vonda and Suzanne (pseudonyms). The purpose of the article was to get ideas about whether or not prospective elementary school teachers (PSTs) can learn to provide better explanations of mathematical concepts during their teacher preparation.

As noted by the authors, since this is only a case study the article cannot actually answer this question in any definitive way. There is no data, no attempt to design a study that could test a hypothesis, and the selection of the four students discussed in the paper was the opposite of random. As such, this article does not, and cannot, be used in support any of Koch's claims, since a paper of this sort should not be used to draw causal inferences.

However, if one were to use some of the tentative ideas the authors offer in the closing sections of the paper, one would have to conclude the *opposite* of what Koch has suggested regarding the importance of mathematical knowledge. From the conclusion, where the authors discuss Vonda, the only one of the four PSTs without an academic high school math class (she took Business Math) and the only one of the four PSTs who did not take at least one calculus course at university they wrote this [our emphasis]:

> *"Vonda's case also highlights the importance of supporting PSTs in both developing high-leverage practices—practices that have leverage for student learning (cf. Lampert 2010)—**and enhancing their content knowledge. This is because limitations in content knowledge might impinge on PSTs' productive engagement with and learning from working on such practices**. Grossman et al. (2009) put this very eloquently, when advocating that high-leverage practices become ''the warp threads of the professional curriculum, **while the knowledge and skill required to enact these practices ... constitute the weft**''* (p. 277)."

Vonda was repeatedly described in the paper as giving incorrect answers and incorrect, incoherent explanations; she was often unable to give any explanation at all. The two courses offered by the education professor in this study did not correct her content knowledge deficiencies. Put succinctly, (and echoing the other Ball papers), the authors repeatedly state versions of the obvious fact that teachers cannot teach math without math content knowledge.

Below are some additional details about, and comments on, this paper.

The two courses the PSTs took, offered by the first-named author through the faculty/department of education, are described by the authors as one math methods class and one math content class; both included 13 three-hour classes offered over approximately 7 months. It should be noted that the content class was not really a proper math content class that could have covered very much material, and it was not offered by a math department. It seems that a main focus of the "content" class was on improving students' abilities to explain a limited number of math ideas to elementary school children. For example, it is clear that a minimum of one of these thirteen classes was devoted to developing classroom explanations for division by fractions, from which it can be concluded that only a very small fraction of the K-8 curriculum could have been covered in this content class. A student entering the class without the prerequisite content knowledge of the full K-8 math curriculum would therefore have left the class without most of this knowledge. This was the case with Vonda.

Regarding the math content knowledge of the students, Nathan and Kimberley had modest backgrounds in math, with six and five university-level math courses (including statistics in the case of Kimberly) respectively; Suzanne had Calculus I from university (and precalculus from high school). Vonda had only business math from high school and Algebra I and II from university. Algebra I and II are *high-school level* math classes that come *before* precalculus; so, in total, Vonda had *at most* the equivalent of Manitoba grade 12 Applied Math 40S or grade 11 Precalculus 30S. No information is given about the students' academic performance in these classes; they may have been A students or D students, but we are not told this.

Koch's position appears to be that Manitoba K-8 teachers do not need more than Vonda's level of content knowledge.

With the exception of Vonda, all students were able to give correct answers to the mathematical questions they were asked as part of this study. Mostly though, PSTs were evaluated on their ability to explain division by a fraction–once before the classes began and once after they had ended. These evaluations were done only by the class instructor and were thus highly subjective; it is our impression that the author brought his own biases into these evaluations. The PSTs' explanations were never presented to children or tested by evaluating children's performance. At the beginning of the study, Nathan and Kimberley gave correct answers to, and mathematically correct visual representations of, the division-by-fractions questions, (though the instructor found their verbal explanations unclear, especially in Kimberley's case); Suzanne obtained a correct answer, but could not initially provide a visual explanation; Vonda was confused about how to answer the questions and gave incorrect, incoherent explanations and incorrect visual representations. After taking the two classes, the author found that Nathan's explanations had improved quite a bit, Kimberley's explanations had improved marginally, and Suzanne's explanations had improved markedly. Vonda's explanations were assessed to have improved marginally but, according to the authors, due to her poor content knowledge her explanations of fraction division never improved very much. (The course instructor did not seem to like Kimberly, who complained about the class moving too slowly. Our impression is that the instructor may have exhibited some bias against Kimberly, her mathematically correct explanations, and her desire to move on. The instructor seemed to feel that Kimberley had a bad attitude.)

The authors concluded that Nathan and Suzanne did seem to learn something from these two (and perhaps other) classes about how to improve their explanations of mathematical content to children.

Conclusions

If one makes a choice of four prospective elementary school teachers, one would have to expect that each would have different strengths and weaknesses. It seems though, that although there might be some variation in their effectiveness, each of Nathan, Kimberly and Suzanne could capably teach elementary school math; Vonda, with her weak content knowledge, could not.

As noted by the authors, this article describes a case study of four, non-randomly selected students. There is no data, no attempt to design a study that could test a hypothesis, and the selection of the four students discussed in the paper was not at all random. Therefore, the article does not, and cannot, be used in support of any of Koch's claims.

However, it is impossible to imagine how Koch could have read this paper and concluded that math content knowledge is not important. Echoing the other Ball papers, the authors repeatedly state versions of the obvious fact that teachers cannot teach math without math content knowledge. The courses UW MATH-2903/04 or UM MATH-1080/90 are designed to ensure that early and middle years teachers obtain the math content knowledge that prospective elementary school teachers need, according to the statements made by the authors in this paper.

Copur-Genturk et al. (2013)

Copur-Gencturk, Y. & Lubienski, S.T. (2013). Measuring mathematical knowledge for teaching: A longitudinal study using two measures. *Journal for Mathematics Teacher Education, 16,* 211-236.

Analysis: ***Does not support the claims*. *Study is irrelevant to the claims and there are serious issues with experimental design.***

- Participants were practicing teachers enrolled in a Master's program. They all held teacher certification before enrolling in the program.
- The math content course is described as follows: "Instead of focusing on mathematical knowledge specific to elementary school teaching, the course was an adapted version of a popular course that fulfilled a general university mathematics requirement."
- Again, the math content course in this study is not of the sort offered to pre-service K-8 teachers in Manitoba. In fact, the math content included in the hybrid course in the study more closely resembles the courses we offer than the math content course that was taught by the math department in this study. Indeed, they describe the content portion of the course as "designed to deepen teachers' knowledge of mathematics content specifically relevant to elementary and middle school teaching," which is precisely the intent of UW MATH-2903/2904 and UM MATH-1080/1090.
- Listing this study as evidence to support the claims made is an example of applying the conclusions out of context, and under the faulty premise that K-8 pre-service teachers are required to take standard undergraduate math courses.
- In addition to the study being applied out of context, there are also serious issues with the methodology and design of the study (some are noted by the authors themselves), making it irresponsible to draw causal inferences. It should not be used to inform public policy.

Summary: The study involved testing 24 teachers using two assessments: Learning Mathematics for Teaching (LMT) and Diagnostic Teacher Assessments in Mathematics and Science (DTAMS). They first took a hybrid course, which was a mix of math methods and math content for elementary teachers. The hybrid course was taught by one of the authors. After completing the hybrid course, they took a math content course, which was not taught by one of the authors and is nothing like UW MATH-2903/2904 or UM MATH-1080/1090. The LMT captured better gains during the hybrid course and the DTAMS captured better gains during the math content course. Regarding the methodology, the authors list several limitations on Pg. 131–132.

Loucks-Horsley et. al (2010)

Loucks-Horsley, S., Stiles, K.E., Mundry, S., Love, N. & Hewson, P.W. (2010). *Designing professional development for teachers of science and mathematics.* Corwin.

__Analysis:__ ***Does not support the claims. This book is about professional development. As well, Koch included quotes in her research synopsis without including the full context and fails to include statements the authors make about the importance of content knowledge taught by content experts.***

In her research synopsis, Koch quotes one sentence out of context from Pg. 15 of this book: "Excellent science and mathematics teachers have a very special and unique kind of knowledge that needs to be developed throughout their professional learning experiences."

Koch then concludes: "Here again, we see that the mathematics knowledge that makes a difference for children in classrooms is qualitatively different than the mathematics taught in many undergraduate mathematics courses."

The authors did not conclude this, nor did they make recommendations that pre-service teachers should not take undergraduate math courses.

As well, Koch neglects to mention the rest of that paragraph, where the authors identify content knowledge as ***critical*** and make it clear that ***pedagogy courses are not enough***. Koch, on the other hand, has argued publicly that pre-service K-8 teachers need only take courses from Faculties of Education, which, by definition, are pedagogy courses.

The entire paragraph is as follows:

"Excellent science and mathematics teachers have a very special and unique kind of knowledge that needs to be developed through their professional learning experiences. Pedagogical content knowledge, that is, knowing how to teach specific scientific and mathematical concepts and principles to children at different developmental levels, is the unique province of teachers and must be the focus of professional development. ***Knowledge of content, although critical, is not enough, nor is knowledge of general pedagogy.*** *There is something more to professional development for science and mathematics teachers than generic professional development opportunities are able to offer.*"

Quote from the book that directly contradicts Koch's point of view

Although this book focusses on professional development, which is not relevant to topic at hand, it's worth noting that Koch also neglects to mention that the authors praise the benefits of content courses offered by content experts as one means of professional development (p. 183).

"*Courses are facilitated or taught by a content expert.* ***Since the intent of this professional learning strategy is to deepen teachers' content knowledge, the leader of the course needs to have extensive understanding of the content.*** *Ideally, the facilitator is also able to connect the content that is the focus of the course to other topics and concepts within the discipline*."

We note that nowhere in this book do the authors argue against pre-service K-8 teachers taking math content courses from mathematics departments as part of their degree.

Davis & Simmt (2006)

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics,61*, 293-319.

Analysis: ***Does not support the claims. Study is not experimental, and the authors themselves note that they're "not yet at a point in the research that we are able to make claims about the relationship between teachers' mathematics-for-teaching and their pedagogical effectiveness."***

In her research synopsis, Koch zeros in on two sentences from the introduction of this paper, noting that they questioned "the assumption that courses in formal mathematics are vital to effective teaching" and cited studies that show that there is "at best a weak relationship between the courses taken by teachers and their students' performances on standardized examinations."

Regarding the first quote, the authors are discussing courses in "formal mathematics," by which they mean the typical math undergraduate course for majors, students in science et cetera (like calculus). Again, we have specialized courses for elementary teachers, so this is an example of Koch attempting to provide evidence against a faulty premise that Manitoba pre-service K-8 teachers are required to take standard undergraduate mathematics courses.

Regarding the second quote, the authors cite Begle's work, which is already discussed in the first section of this document; it does not support Koch's claims and has serious design flaws. The second article they cite is Monk's work, which is discussed later in this document and also does not support Koch's claims.

Summary: This paper is mixture of two things: 1) a report on discussions among a group of 24 teachers on their understanding of multiplication, held during a one-day seminar facilitated by the authors, and 2) some theorizing on how the theory of complex systems might be used as a framework for analyzing how teachers come to understand mathematical concepts. It contains no experimental data, no measures of the teachers' mathematical knowledge or indications of their prior university training, and no measures of their pedagogical effectiveness. The authors discuss four aspects of teachers' *mathematics-for-teaching*, namely, "mathematical objects," "curriculum structures," "classroom collectivity," and "subjective understanding" and "**conjecture** [my emphasis] (1) that a particular fluency with these four aspects is important for mathematics teaching and (2) that these aspects might serve as appropriate emphases for courses in mathematics intended for teachers" (p. 293, Abstract). However, they conclude by saying, "**Unfortunately, we are not yet at a point in the research that we are able to make claims about the relationship between teachers' mathematics- for-teaching and their pedagogical effectiveness**" (p. 317).

Davis & Renert (2013)

Davis, B. & Renert, M. (2013). *The math teachers know: Profound understanding of emergent mathematics.* Routledge.

Analysis: ***Does not support the claims.*** At no point do the authors of this book argue that K-8 pre-service teachers should not take specialized math content courses for elementary teachers of the type Manitoba math departments provide. In fact, our reading of the authors' work reveals that they appear to argue in favour of the need for K-8 teachers to have deep content knowledge of the material they need to teach. At one point ***the authors express that they disagree with colleagues who claim that undergraduate math courses are unhelpful for K-8 teachers.***

Regarding K-8 teachers and undergraduate courses, they again cite Begle (1972, 1979), whose work is discussed at the beginning of this document and does not support Koch's claims.

Quote from the book that directly contradicts Koch's point of view

The authors write that they disagree with math educators who claim that mathematics undergraduate courses are unhelpful, so it may be that Koch is misrepresenting their opinion. As well, this indicates that even the opinion that pre-service K-8 teachers should not be required to take standard undergraduate mathematics courses is just that–an opinion–and not one that is agreed upon by all experts in the field. (We remind the reader that Begle's work has been critiqued at the beginning of this analysis–it does not allow for causal inferences.)

> "*However, the thinking behind these practices is problematic. As Begle (1972, 1979) showed, there is little or no correlation between teachers' college credits in mathematics and their students' performance. Even so, the conviction persists that future mathematics teachers should slog through the same mathematics courses as future physicists, engineers, and computer scientists. Is this a good idea?*
>
> *Many have argued that it is not. To be honest,* ***we're not so sure as some as our colleagues on the matter. Both of us have studied considerable amounts of mathematics at the university level, and both of us routinely draw on insights gleaned from university courses to inform our teaching at the elementary and secondary school levels. It has been our experience that the time spent in university mathematics courses matters. True, it is not always clear how or why such study is important or relevant – but that's precisely the point.***" (Pg. 5)

Gaulin (2016, 1977)

Gaulin, C. (2016, 1977). Innovations in teacher education programs. In P. Liljedahl et al. (Eds.) Canadian Mathematics Education Study Group Special Issue: 40 Years of CMESG (pp. 43-52).

Analysis: ***Irrelevant to the claims. Does not support the claims.*** At no point in the paper does Gaulin argue that future teachers of mathematics at any level (elementary or higher) should not take courses offered by math departments, nor does Gaulin argue that K-8 teachers shouldn't be required to take math content courses.

Summary: The paper goes over the organization of the teacher training programs at the time and discusses issues and some innovations with the aim of improving the organization of such programs. The paper is specifically about training of teachers of mathematics.

The paper starts by reviewing the organization of the traditional teacher training programs, breaking this into Pre-service Teacher Training (PRESET) and In-service Teacher Training (INSET). In PRESET, the future teachers have their basic training, including subject training. INSET refers to training that in-service teachers can receive. At the time of publication, according to Gaulin, the INSET was often in the form of necessary retraining and updating (say, if there was a change in the curriculum and teachers needed to learn something new as a result), or activities that gave the participating teachers an opportunity to eventually gain an increase in salary. PRESET was almost always provided by universities, while INSET was provided partly by universities and partly by other organizations (e.g. school boards, teacher associations, ministries of education). Gaulin then discusses a few issues with this organization of teacher training, particularly with the PRESET stage: (1) a lack of integration and of the different topics and courses trainees learn (i.e. not connecting them properly), (2) a gap and imbalance between theory and practice, and (3) a lack of coordination between the mathematics and mathematics education component of PRESET programs.

Gaulin then discusses some innovations, proposals and possible trends for reorganizing teacher training. In particular, he discusses the possibility of taking "continuing education" as the conceptual framework for PRESET and INSET, essentially meaning the teachers can continue to add to their knowledge throughout their career, in a way distinguishable from how things were at the time, as follows. He argues that this means that PRESET *"should no longer be thought of nor organized as if it were aimed at preparing a teacher for a whole career"*. He then argues that *"if properly organized, INSET should allow any inservice teacher to eventually take any course he [the teacher] may have "missed" and which was optional in PRESET."* The difference between this and the "traditional training" is in the way INSET is used: instead of workshops or short trainings to obtain salary increases, teachers can use INSET to take more university-level math courses. Gaulin then reviews a new approach to INSET taken at the time in Quebec (PERMAMA programme) and ends the paper with a few further thoughts on the future role and responsibility of universities in teacher education. His proposal of using INSET opportunities to add to one's mathematical knowledge (possibly by taking new courses) certainly cannot apply to subjects the teacher will be expected to teach right from the start. For instance, a teacher who is expected to teach fractions to students from the start of their career cannot try to get comfortable with fractions as a part of an INSET!

Monk (1994)

Monk, D.H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*(2), 122-145.

Koch also provided research articles related to preparation of secondary teachers, one of which is the above article, authored by Monk. Further down in her synopsis, she notes that requiring undergraduate mathematics courses for admission to after-degree B.Ed. programs "may also have some drawbacks for prospective high school teachers." This statement is not supported by the articles Koch provided.

Koch notes that "importantly Monk's study also shows that courses in mathematics and science pedagogy also contributed positively to achievement of high school students." She quotes from Monk's article that "on several occasions [math pedagogy courses] had more powerful effects than additional preparation in the content area."

__Analysis__: ***Data provided in the article does not support Koch's claims. In particular, the first of the above quotes in the context of the Monk article seems to refer to data that was not statistically significant at the 5% level. With respect to the third of the above quotes–in which it is asserted that math pedagogy courses had more powerful effects than additional preparation in the content area–this conclusion cannot and should not be drawn from research reported on in this article.***

Note that this paper is irrelevant to the claims in Koch's Winnipeg Free Press article, primarily because it includes Grades 10 & 11 teachers. Our best guess is that it was included in attempt to argue for more pedagogy courses and fewer math content courses for secondary teachers, but the data in the article doesn't support that either.

It may be relevant to note that some other education authors have attempted to use Monk's article to claim that taking more math courses offers diminishing returns for teachers or that an increase in undergraduate math coursework for teachers is negatively correlated with student achievement. However, the statistic referred to in this case is not statistically significant at the 5% level, nor has this finding been replicated by any other study, to the best of our knowledge. In our view, it is therefore irresponsible to make that claim without at least noting the lack of statistical significance as well as the other issues with the study design.

Some of the design flaws are mentioned by Monk in the paper, but a 2003 review of teacher characteristics, authored by Wayne & Youngs, noted that this article by Monk was excluded because the authors did not control for socioeconomic variables. See:

Wayne, Andrew & Youngs, Peter. (2003). Teacher Characteristics and Student Achievement Gains: A Review. Review of Educational Research. 73. 89-122. 10.3102/00346543073001089.

In our opinion, there are other design flaws and omissions, while we'll discuss below, that make it inappropriate to draw inferences from this study. For example, the author does not discuss the validity of the regression model's goodness of fit, making its predictability questionable.

We also have concerns about drawing conclusions about teacher characteristics from 1994 (or earlier) and their impact on student achievement when more recent studies find positive correlations between secondary teachers' completion of undergraduate math courses (or having math degrees) and student achievement; we mention some at the end of this document.

__Summary:__

This research used the Longitudinal Study of American Youth (LSAY) to try to measure the effect of a teacher taking university math and science courses, and/or math education/science education courses, on math and science test scores for sophomore (Grade 10) and junior (Grade 11) students. The author reports the following findings:

- Monk states his analysis "reveals positive relationships between the number of undergraduate mathematics courses in a teacher's background and improvement in students' performance in mathematics."

- The improvement in student test scores, per university mathematics course taken by the teacher, is quantified. He first considers junior courses: "(A)n increase of one mathematics course for a teacher with a modest level of mathematics training (i.e., five or fewer mathematics courses) is associated with a 1.2% increase in the student's mathematics test score, when the increase is figured at the mean of the dependent variable. For sophomores the magnitude is considerably smaller. The same one course increase is associated with a 0.2% increase, again when the increase is figured at the mean of the dependent variable."
- Monk reports that math education courses are positively correlated with performance also: "For the undergraduate mathematics education courses. the impact of an additional course is on the order of a 0.4% increase in test score performance."
- The author characterizes the undergraduate math education courses as having more impact than the undergraduate math courses; however, his figures support this for Grade 10, but they *support the opposite conclusion for Grade 11*.

  -The regression coefficients for **undergraduate mathematics courses** in the junior year models are **0.81** and **0.77** (with standard errors of 0.33). These effects are statistically significant with $p<0.01$, indicating that taking more undergraduate math courses was positively associated with higher math achievement scores.

  -The regression coefficients for **undergraduate math education courses** in the junior year models are **0.29** and **0.27** (with standard errors of 0.14 and 0.13). These effects are statistically significant with $p<0.02$, *but the magnitude of the effect is smaller than that of undergraduate math courses*.

  -This indicates that taking undergraduate math education courses also improves math achievement scores for Grade 11 students, but the impact is less pronounced compared to taking undergraduate mathematics courses.

- The study also considered the effects of taking >5 undergraduate math courses. However, the regression coefficients for >5 undergraduate math courses are not statistically significant at the 5% level for either Grade 10 or Grade 11 students. At the junior level, for example, they are **-0.60** and **-0.62.** The standard errors are relatively large (**0.38**), and they are *not statistically significant at the 5% level.*

Regarding Koch's statement that "on several occasions [math pedagogy courses] had more powerful effects than additional preparation in the content area." This is a direct quote from the article, so perhaps Koch can be granted some leniency here. However, our opinion is that one should examine the data provided in the article and consider the statistics involved before drawing strong conclusions. As noted above, the data seems to support this assertion for Grade 10, but it supports the opposite conclusion for Grade 11. As well, there are other important considerations that should be noted:

- The math teachers included in the study had, on average, 7.68 undergraduate math courses (SD 4.29) and 1.91 undergraduate math education courses (SD 2.33). It's therefore impossible to conclude anything about the impact of taking more mathematics pedagogy courses since most teachers in the study had not taken many mathematics pedagogy courses. Indeed, while data is reported for >5 math undergraduate courses (again, the result is not statistically significant at the 5% level), no such data is provided for math pedagogy courses.

- This study included students in Grades 10 and 11 only. Manitoba high school teachers also teach Grade 12 and often teach subjects like AP/IB or dual-credit Calculus. It is reasonable to conclude that even more math undergraduate courses are required to effectively teach these higher-level secondary courses.

Goulding et al. (2003)

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: Its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education, 6*(4), 361-393.

Analysis: ***Study is not an experimental study and is irrelevant to the claims.***

Summary: This paper reports on a survey that was given to students in the UK who were studying to become teachers by taking a one-year after-degree education program. The paper authors asked students how they had found their undergraduate math degree. A certain fraction of students found their undergraduate degree difficult, and resented both the material, and the delivery.

It's unclear why Koch included this obscure study. Our best guess is that Koch is trying to suggest that mathematics professors in Manitoba are not good at teaching undergraduate mathematics courses. That's a difficult notion to back up, considering our departments have many professors who have won teaching awards, both at the institutional level and nationally.

There's no reason to believe that undergraduate mathematics programs or professors in the UK in 2003 are similar to math programs and professors in Manitoba math departments in 2024, so to include this study as supposed evidence of anything is bizarre.

Note on the remaining papers that were analyzed:

Koch included a section on admission requirements and provided a list of non-empirical papers. Most of these are opinion pieces and do not support any claims made by Koch. Moreover, one of the articles recommends that K-8 teachers be required to take 6 cr hrs of math content courses, which Koch neglects to mention. We don't see these papers as particularly relevant to the claims in the Free Press article, but we read and analyzed them, nonetheless.

Caskey, Peterson & Temple (2001)

Caskey, M.M., Peterson, K.D., & Temple, J.B. (2001). Complex admission selection procedures for a graduate preservice teacher education program. *Teacher Education Quarterly, 28*(4), 7-12.

***Analysis:* Irrelevant to the claims. We see no direct link between the article and the claims made by Koch.**

Regarding the merit of the research and the validity of its conclusions, it is important to keep in mind that the study was done on ONE American university's applicants.

Summary: This paper analyses the data associated to the admission process of a one-year graduate teacher preparation program at an urban university in the United States. The population consists of 141 applicants, and the authors study the correlation between some factors considered in the admission process, including among others the applicants' undergraduate GPAs, letters of recommendation, personal statements, and standardized test scores on a test of basic academic skills (reading, writing, mathematics), and the admission acceptance. For the admitted students, the authors also study the "predictive validity" of some of these selection variables by considering students' success rates at the end of the program. The success rate is measured based on ratings given by the faculty members teaching in the program to students.

Childs et al. (2011)

Childs, R.A., Broad, K., Gallagher-Mackay, K., Sher. Y., Escayg, K. & McGrath, C. (2011). Pursuing equity in and through teacher education program admissions. *Education Policy Analysis Archives, 19*(24), 2-19.

Analysis: ***Irrelevant to the claims. As well, this is a case study. The authors do not measure in any meaningful way how the admission study processes described in the paper will result or have resulted in improvement to students' numeracy or literacy.***

Summary: This case study investigates considerations of equity in teacher education admission *processes and procedures* through the years 2002-2003 to 2009-2010 in a large teacher's training program in Ontario. The authors do this *"through document analysis and structured interviews with ten past or current members of the admissions committee."* The authors are interested in two types of equity, namely, what they refer to as equity 1) *in* admissions, and 2) *through* admissions. The former is about equity in the admission process and its outcome (who is admitted in the program), and the latter is about the future students of the teachers who will be accepted and trained in the program.

The authors say their motivation for the study was that they thought that despite making the changes to the admission procedures in response to the admission policies 'calls for equity, an operationalization of equity continued to be elusive. The authors wondered if it was possible that there were inherent contradictions in the pursuit of equity. The authors argue in their paper that pursuit of equity in and through admissions do not lead to the same decisions.

Childs & Ferguson (2015)

Childs R. A., Ferguson A. K. (2015). Changes in, to, and through the initial teacher education program admission process. In Thomas L., Hirschkorn M. (Eds.), *Change and progress in Canadian teacher education: Research on recent innovations in teacher preparation in Canada* (pp. 420–440). Canadian Association for Teacher Education.

Analysis: ***This is a discussion article, not a research article. Further, Koch neglects to mention a reference in the article to recommendations made by the Sub-Committee of the Mathematics***

Education Forum of The Fields Institute for Research in Mathematical Sciences that pre-service teachers be required to "take at least one undergraduate course in mathematics, but preferably two."

Summary: The authors discuss the process of admitting teacher candidates to initial teacher education programs and they discuss several problems that arise. They argue that considering what problems admission processes are called upon to solve is important and changes over time.

Quote from the article that directly contradicts Koch's point of view

Note the following portion, which refers to a recommendation that pre-service teachers take math content courses (Pg 426-427). This directly contradicts Koch's recommendations.

"*Despite the fact that standardized test results in mathematics for Ontario are showing promising results, these results remain somewhat stagnant over time and continue to point to a vast number of students still being left behind. While we have some exceptional teachers of mathematics at the P/J/I levels, there is no consistency in their depth of understanding of the material they teach – the depth needed for students to achieve at the highest levels. (Kajander, Kotsopoulos, Martinovic, & Whitely, 2013, p. 63)*

The subcommittee recommended to the Ontario College of Teachers (OCT) that "Teachers admitted to Primary/Junior/Intermediate teacher education must have at least one undergraduate course in mathematics, but preferably two" (p. 62)."

See https://www.fields.utoronto.ca/journalarchive/FMEJ/81-41.pdf for the Fields Institute report.

Lukas & Samardzic (2015)

Lukas, M. & Samardzic, D. (2015). Admission requirements for teacher education as a factor of achievement. SGEM 2015 International Multidisciplinary Scientific Conferences on Social Science and Arts Conference Proceedings. https://files.eric.ed.gov/fulltext/ED560901.pdf

Analysis: ***Article is irrelevant to the claims.***

Summary: The authors compare admission procedures for candidates to teacher education programs in four developed countries (Finland, Japan, Korea and Singapore) and four developing countries (Croatia, India, Russia, Turkey). The four developed countries each require a candidate to pass an interview for admission to a teaching program whereas no such requirement exists in the four developing countries. Since these four developed countries performed better on the PISA exams than the four developing countries, the authors believe that the inclusion of an interview to assess the attitudes and motivation of candidates as part of the admissions process leads to more effective teachers.

Note: the authors do not argue for lowering academic requirements for admissions to teaching programs.

Holden & Kitchen (2016)

Holden, M. & Kitchen, J. (2016). Evolving practices: Admissions policies in Ontario teacher education programs. *Canadian Journal of Education, 39*(4), 1-28.

Analysis: ***Article is irrelevant to the claims.***

Summary: This paper is a scan of the admissions requirements for teacher education programs at 15 Ontario universities, particularly with regard to: 1) which institutions use academic averages as a measure of cognitive skills; 2) which institutions use non-cognitive written statements and references; and 3) which institutions have explicit equity-based admissions policies. It is concerned with admissions policies only, not about teacher training, and has nothing to do with mathematics education. The words 'math' or 'mathematics' do not appear in the document.

McDuffie et al. (2024)

Roth McDuffie, A., Slavit, D., Goldhaber, D., Theobald, R., & Griggs, N. (2024). Attention to Equity in Teacher Education Admissions Processes. *Journal of Teacher Education*, *75*(3), 275-291. https://doi.org/10.1177/00224871241230306

Analysis: ***Article is irrelevant to the claims.***

Summary: The authors investigated 31 K-12 *mathematics and science* teacher preparation programs to determine how the deal with diversity, equity, inclusion, and social and racial justice (DEIJ). There is no mention of undergraduate math content courses in the document and they do not argument for removing subject-area requirements for admission.

CONCLUSIONS AND IMPLICATIONS FOR OTHER SUBJECT AREAS

As Dr. Narad Rampersad noted in a letter to the editor in the Winnipeg Free Press (*Letters, November 18*), in reference to Koch's dubious claims, "extraordinary claims require extraordinary evidence." It is our view that Koch has not provided *any* relevant evidence to support her extraordinary claims, let alone *extraordinary* evidence.

We hold the commonsense view that teachers must have a solid understanding of the subject areas they will be required to teach. We also support the standard practice in universities that math content courses should be taught by subject-area specialists in mathematics departments.

The Manitoba government's sweeping changes–eliminating all subject-area requirements for teacher certification–break with a long-standing professional expectation rooted in common sense. The decision defies logic and reason, and to date, Manitobans have received no evidence that these changes will not lower the quality of education for Manitoba children. In fact, common sense suggests they will lower educational quality.

Our review highlights significant discrepancies between Koch's statements and the evidence from the references she provided. The potential impact of these inaccuracies is profound, particularly as they pertain to shaping policy decisions that impact teacher preparation and student outcomes in Manitoba.

In her research synopsis Koch wrote that "I have included some of the research which I have found compelling as a mathematics education researcher and teacher educator over the past 20 years. I include research that *should inform policy decisions*." Given our review of the research she included, this statement raises serious concerns.

Statements made by the Acting Minister of Education and Early Childhood Learning in the Manitoba Legislative Assembly on November 22, 2024, suggest that these unsupported math education research claims have informed policy decisions, which is alarming. It is crucial to prioritize transparency and consult broadly in educational policymaking to ensure decisions serve the best interests of teachers, students, and the wider community.

We have focused on specific facets of mathematics education here, but the amendments to the *Education Administration Act* raise broader concerns about the qualifications of teachers more generally. The removal of any requirement for in-depth study at the university level (e.g., for a major) and the elimination of breadth requirements for English/French, science and history/geography diminish important academic foundations for effective teaching. This raises questions about the academic competencies of individuals entering the teaching profession in the future, which has implications for informed teaching, not only of mathematics, but all subjects that benefit from rigorous study, including other critical ones like reading, writing, science, history, and geography.

To address the issues identified in this report, we urge Dr. Koch to retract her Winnipeg Free Press article, and we call on the Manitoba government to consult more broadly and exercise greater caution when presented with research in education.

Appendix A: Samples of relevant articles omitted by Koch

To demonstrate that there are relevant articles that contradict Koch's claims, we include a sample of articles that were omitted by Koch.

Metzler & Woessmann (2012)

Johannes Metzler, Ludger Woessmann, The impact of teacher subject knowledge on student achievement: Evidence from within-teacher within-student variation, Journal of Development

Economics, Volume 99, Issue 2, 2012, Pages 486-496, ISSN 0304-3878, https://doi.org/10.1016/j.jdeveco.2012.06.002.

We include this paper as an example of a study on teacher characteristics that uses a more rigorous design. The authors rightly note that:

"*identifying the causal effects of teacher characteristics on student achievement econometrically is a difficult task. Problems of unobserved student and teacher characteristics and of non-random selection into classrooms are likely to bias the estimates available in the literature. If such omitted variables and selection processes are correlated with the achievement of both teachers and students – as is quite likely in such cases as teacher motivation and pedagogical skills, student effort and ability, parental choice of schools and classrooms, and student placements into classrooms – the available conventional estimates will not capture the true effect of teacher knowledge on student outcomes.*"

The authors proposed a new method to examine the effect of teacher subject knowledge on student achievement,

"*drawing on the variation in subject matter knowledge of teachers across two subjects and the commensurate achievement across the subjects by their individual students. By restricting the sample to students who are taught by the same teacher in both subjects, and in schools that have only one classroom per grade, this identification approach is able to circumvent the usual bias from omitted student and teacher variables and from non-random selection and placement. Furthermore, possible bias from measurement error in teacher subject knowledge was addressed by reverting to psychometric test statistics on reliability ratios of the underlying teacher tests*."

The authors found a significant effect of teacher subject knowledge on student achievement for reading and math. The data set consisted of 6^{th} grade Peruvian students and their teachers. They found that

"*A one- standard-deviation increase in teacher subject knowledge raises student achievement by about 10 percent of a standard deviation*."

The authors note that the extent to which the results generalize to developed countries remains unknown, but they recommend that

"***The results suggest that teacher subject knowledge should be clearly on the agenda of educational administrators and policy-makers. Attention to teacher subject knowledge seems to be in order in hiring policies, teacher training practices, and compensation schemes**.*"

Wang et al. (2023)

Wang, Xiaofang & Perry, Laura & Malpique, Anabela & Ide, Tobias. (2023). Factors predicting mathematics achievement in PISA: a systematic review. Large-scale Assessments in Education. 11. 24. 10.1186/s40536-023-00174-8.

The authors reviewed 156 studies that investigated multiple factors that affect mathematics performance on international PISA tests. Strict screening procedures were applied to 3144 eligible articles to land on the 156 studies. For example, one criteria was that the article must have been published in the top quartile (Q1) in any discipline in Scimago Journal & Country Rank between 2016 and 2020. They found that one consistent finding was the importance of teacher qualifications. Seven of the 156 studies examined teacher qualifications as a predictor of mathematics performance.

"***Teachers' qualifications tended to be positively associated with math achievement in seven relevant studies****. Six studies found a positive association between higher level of teachers' qualifications and math (e.g., Chiu, 2015), associated with teachers having a bachelor's degree or higher (e.g., Luschei & Jeong, 2021; Orón Semper et al., 2021),* ***a major in math*** *(Carnoy et al., 2016; Kim, 2018), or having qualifications higher than required (Cordero & Gil-Izquierdo, 2018). One study (Liu et al., 2022) found no association, but this may be because teacher qualification was operationalized by combining formal education with years of work experience, which other studies (Carnoy et al., 2016; Cordero & Gil-Izquierdo, 2018) found it was not associated with math achievement*."

Kleickmann et al. (2017)

Kleickmann, T., Richter, D., Kunter, M., Elsner, J., Besser, M., Krauss, S., & Baumert, J. (2013). Pedagogical content knowledge and content knowledge of mathematics teachers: The role of structural differences in teacher education.

This study emphasizes the importance of formal training for both content knowledge (CK) and pedagogical content knowledge (PCK) in teachers. The notion of PCK arose in the work of Shulman, in an article that was summarized earlier in this document.

The authors note that *"Both types of knowledge [CK and PCK] have been shown to affect teachers' instructional practice as well as student learning in the domain of mathematics (Baumert et al., 2010; Hill, Rowan, & Ball, 2005)."*

The article is concerned with the impact of structural differences in teacher education on teachers' CK and PCK.

The authors note that "*There is some consensus and some preliminary evidence for the notion that CK might be a prerequisite for PCK development."*

We include some relevant quotes from the article.

"For example, the authors found that future U.S. teachers whose teacher education programs included rigorous and demanding mathematics courses showed higher mathematical CK than those who attended other programs (Schmidt et al., 2007)."

Figures on page 98 show higher levels of CK and PCK for teachers in training from an academic track. The authors conjecture that content knowledge is a pre-requisite for pedagogical content knowledge.

"*Future academic-track teachers showed higher increases in CK from Year 1 to Year 3 of university education as well as from Year 3 to the induction phase than did future nonacademic-track teachers. At the end of teacher education, the differences between teachers of the academic and non- academic tracks were particularly large.*"

"*Thus, our findings may again point to the importance of CK in the development of PCK.*"

"*In Germany, for example, future academic-track teachers are more likely to opt for advanced mathematics courses in Grades 11 to 13 than future nonacademic-track teachers. Thus, they have more formal learning opportunities to acquire mathematical knowledge before beginning their professional education.*"

"We found support for this hypothesis for CK and PCK. The future academic-track teachers, who have much more formal learning opportunities for CK in the first phase of teacher education, showed considerably higher gains in CK from Year 1 to Year 3 than did future nonacademic-track teachers. Furthermore, the inservice phase, which involves primarily informal learning, does not seem to foster the development of CK and PCK as strongly as the formal and nonformal learning opportunities provided by initial teacher education programs."

"During the university phase of teacher education, prospective academic- and nonacademic-track teachers showed similar differences in PCK from Year 1 to Year 3, although the latter have more formal learning opportunities for the development of PCK during their university studies. We interpreted this finding as a further point to the ***importance of CK in the development of PCK. Higher CK may lead to increased uptake of learning opportunities to acquire PCK, thus compensating effects of the quantity of learning opportunities****."*

"The findings for PCK are consistent with the particularly low performance of Germany's nonacademic track students on the Third International Mathematics and Science Study and PISA assessments—assuming that teachers' PCK is a crucial prerequisite for student learning (Baumert et al., 2010; Hill et al., 2005). Our study provided further evidence that the deliberative formal and nonformal learning opportunities provided in the context of initial teacher education are crucial for the development of teachers' subject matter knowledge. In contrast, informal learning in the form of incidental learning, often referred to as teaching experience, seems to have only a weak effect on the development of teachers' subject-matter knowledge, especially CK."

The findings then seem to highlight the importance of comprehensive and structured teacher education programs in equipping teachers with the necessary skills and knowledge to improve student outcomes. They also don't support any notion that content knowledge can be acquired through teaching experience.

There are also social impacts, and the quote below indicates just how important it is that all teachers receive high quality formal training in CK and PCK:

"Students attending nonacademic-track schools differ from their peers in the academic track not only in their ability and achievement but also in their social and ethnic backgrounds. Consequently, low-achieving students from families with lower socioeconomic status and immigrant families tend to be taught by teachers who are less competent in terms of CK and PCK."

As well, the table on pg 103 shows that advanced courses and interest in math contribute to both CK and PCK.

The authors conclude by noting that *"Our findings highlight the urgent need to improve the preparation of future nonacademic-track teachers with respect to subject-matter knowledge (CK and PCK). Moreover, candidates for the academic track already enter teacher education with better subject-matter knowledge, and our findings suggest that these differences persist or even increase across the teaching career. Thus, in addition to improving teacher education, changes in recruitment and selection processes for teacher candidates could help to raise the quality of instruction and student progress in nonacademic-track schools."*

Goldhaber & Brewer (2000)

Goldhaber, D. D., & Brewer, D. J. (2000). Does Teacher Certification Matter? High School Teacher Certification Status and Student Achievement. Educational Evaluation and Policy Analysis, 22(2), 129-145. https://doi.org/10.3102/01623737022002129

We include this paper to illustrate that the literature on this topic is mixed, with various studies reaching different conclusions. These authors state the following:

"*Consistent with Goldhaber and Brewer (1997a), math students who have teachers with Bachelors or Masters degrees in mathematics are found to have higher test scores relative to those whose teachers have out-of-subject degrees."*

Further, this study found that

"having a degree in education has no impact on student science test scores, and, in mathematics, having a BA in education actually has a statistically significant (at the 10% level) negative impact on mathematics scores of students."

It is important to note for this second result, that statistical significance at the 10% level does not provide sufficient evidence to make definitive conclusions. The study may also be criticized for its experimental design. However, Koch has previously drawn conclusions from studies with similar levels of significance, and weaker designs, so one might wonder why this paper was not included alongside the Monk (1994) paper, for instance.

Appendix B: Preface to Mathematics for Elementary School Teachers by Ricardo D., Cengage, ISBN: 9781133983606.

This textbook is used in UW MATH-2903/2904 and UM MATH-1080/1090.

Preface

My motivation for writing this textbook stems from searching for ways to relate the mathematics in college courses for prospective elementary school teachers to the mathematics in the elementary curriculum.

My goal is to give prospective elementary school teachers a profound understanding of the mathematical content they are expected to know and be able to teach.

My writing has been shaped by reviewing and incorporating the literature and standards from both the National Council of Teachers of Mathematics (NCTM) Standards and Expectations and the Common Core State Standards (CCSS). The standards describe knowledge and skills that students should have, and, in turn, describe what teachers should be able to teach. It was shaped by my experiences teaching prospective elementary school teachers in college classrooms for more than 15 years and by teaching K–6 students during a sabbatical leave, along with hundreds of additional volunteer hours in the elementary classroom during the 6 years of writing this textbook. In addition, the writing was shaped by class testing the material, which gave me the opportunity to incorporate valuable feedback from students and colleagues. These reinforcing factors made it possible to narrow the gap between theory and practice by developing, testing, and refining accessible ways to convey the mathematical content.

As I wrote, researched, and revised the chapters, it was important to answer some key questions.

- What topics are given highest priority by the NCTM Standards and Expectations?
- What recommendations are published in the literature?
- What mathematical strategies promote comprehension?
- What are effective ways to develop the mathematical topics so that prospective elementary school teachers can readily apply them in their own classrooms?
- What examples clarify topics?
- What questions assess understanding and skills?

In this textbook, the quality, quantity, and variety of worked examples and homework questions provide prospective elementary school teachers with opportunities to acquire mathematical knowledge, to develop skills that they can effectively apply in their own classrooms, and to support assessment of the main ideas in the sections, in the spirit of the NCTM standards and CCSS. The worked examples and homework questions reflect my belief that "students learn what they are given opportunities to learn" (Hiebert, 2003).

Enriching the Content through Use of the Standards

The *Principles and Standards for School Mathematics*, published by the NCTM in 2000, represents the most significant and influential collaboration among educators to improve mathematics education at a national level. The relevant NCTM Standards and Expectations appear in the exposition.

The purpose of the NCTM standards is "to ensure quality, to indicate goals, and to promote change" (© NCTM Standards 2011 by National Council of Teachers of Mathematics).

There are two main categories for standards—content standards and process standards—each of which has five subcategories.

Content Standards	Process Standards
Skills, concepts, and understanding that students should acquire for bands of grade levels (Pre-K–2, 3–5, 6–8, and 9–12)	*Ways that students acquire and use knowledge and demonstrate understanding*
Number and operations	Representation
Algebra	Problem solving
Geometry	Reasoning and proof
Measurement	Connections
Data analysis and probability	Communication

The content standards, which include specific goals called expectations, outline the math content that Pre-K–8 students will have to know and understand, which, in turn, indicate the mathematical topics that prospective elementary school teachers will have to understand and teach. The five NCTM process standards highlight ways in which teachers present content; students learn content; and students can demonstrate factual, conceptual, and procedural understanding. "The five process standards are drawn from extensive research on human cognition and mathematics. It is our job as teachers to help students learn how to use these processes appropriately to develop the mathematical knowledge described in the content standards" (Zemelman et al., 2005, p. 112).

REPRESENTATION
PROBLEM SOLVING
REASONING
CONNECTION
COMMUNICATION

I highlight the process standards throughout the textbook in the exposition, worked examples, and homework questions. A colorful icon, which includes the names of the applicable processes, is presented in the margin throughout the book to tie the following process standards to the content.

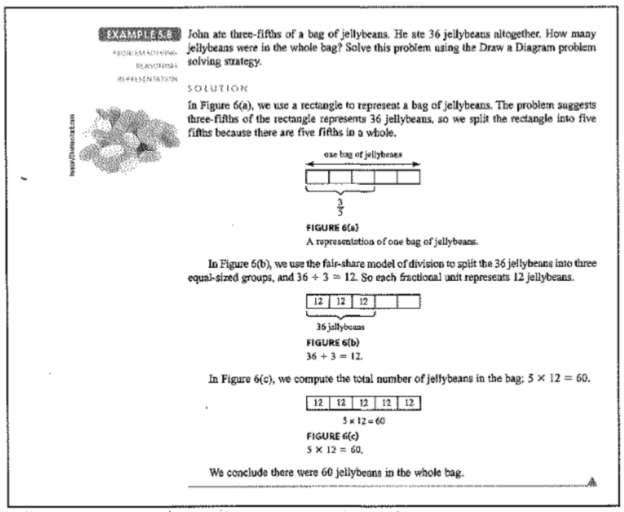
Representation *is the display of mathematical content using language, tables, algebra, diagrams, and symbols in contextualized situations.* These tools help math instructors teach in ways that make mathematics more accessible, relevant, and enjoyable for prospective teachers. They help prospective teachers "articulate, clarify, justify, and communicate their reasoning to others" (Woleck, 2001) not only in ways they will teach their own students but also in ways they will expect their own students to reason and solve problems. Students with strong representational skills achieve more success in mathematics, because representations make mathematical ideas more comprehensible, spark thought, narrow the gap between abstract and concrete ideas, and help students see connections between mathematical ideas. "Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understanding of one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling" (NCTM, 2000, p. 67).

Problem solving *challenges prospective teachers to apply their knowledge in unfamiliar situations.* It helps build and extend their mathematical knowledge and reasoning abilities.

Reasoning and Proof *form a way of thinking, justifying, and making sense of mathematics and providing an explanation (a proof) in a logical and convincing manner.* Types of reasoning include inductive reasoning, deductive reasoning, algebraic reasoning, additive reasoning, multiplicative reasoning, proportional reasoning, probabilistic reasoning, and geometric reasoning.

Connections *involve recognizing relationships between and among topics and are an indicator of understanding.* Students should be able to explain, for example, how the four arithmetic operations are related; how division and fractions are related; how fractions and decimals are related; how the array model of multiplication and the formula for the area of a rectangle are related; how number systems (counting numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers) are related; and how to use diagrams to solve classic algebra, ratio, and percent problems. "When students can connect mathematical ideas, their understanding is deeper and more lasting" (NCTM, 2000, p. 64).

Communication *involves expressing written and verbal explanations in a clear and organized manner and supporting the explanations with diagrams and appropriate mathematical notation, symbols, and vocabulary.* The explanations flow in a logical order within the textbook, which builds the habit among students of communicating in an orderly way.



EXAMPLE 6.6 John ate three-fifths of a bag of jellybeans. He ate 36 jellybeans altogether. How many jellybeans were in the whole bag? Solve this problem using the Draw a Diagram problem solving strategy.

SOLUTION

In Figure 6(a), we use a rectangle to represent a bag of jellybeans. The problem suggests three-fifths of the rectangle represents 36 jellybeans, so we split the rectangle into five fifths because there are five fifths in a whole.

FIGURE 6(a)
A representation of one bag of jellybeans.

In Figure 6(b), we use the fair-share model of division to split the 36 jellybeans into three equal-sized groups, and $36 \div 3 = 12$. So each fractional unit represents 12 jellybeans.

FIGURE 6(b)
$36 \div 3 = 12$.

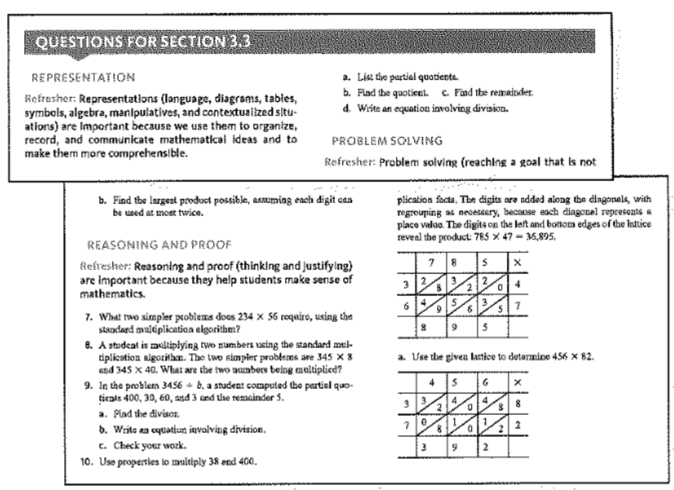
In Figure 6(c), we compute the total number of jellybeans in the bag; $5 \times 12 = 60$.

FIGURE 6(c)
$5 \times 12 = 60$.

We conclude there were 60 jellybeans in the whole bag.

To help students comprehend the five process standards, each worked **Example** illustrates concepts and techniques and an icon indicates which of the five process standards are emphasized in the example.

Many **End-of-Section Questions** are grouped according to the process they emphasize to help students organize their work and to help them understand how processes affect their choice of problems in their own classroom. In each section exercise set, a **Refresher** reminds students of each process standard. The run of exercises that follows gives prospective teachers an opportunity to explore the particular process standard and relate that standard to mathematical content.



QUESTIONS FOR SECTION 3.3

REPRESENTATION

Refresher: **Representations (language, diagrams, tables, symbols, algebra, manipulatives, and contextualized situations) are important because we use them to organize, record, and communicate mathematical ideas and to make them more comprehensible.**

a. List the partial quotients.
b. Find the quotient. c. Find the remainder.
d. Write an equation involving division.

PROBLEM SOLVING

Refresher: **Problem solving (reaching a goal that is not**

b. Find the largest product possible, assuming each digit can be used at most twice.

REASONING AND PROOF

Refresher: **Reasoning and proof (thinking and justifying) are important because they help students make sense of mathematics.**

7. What two simpler problems does 234×56 require, using the standard multiplication algorithm?
8. A student is multiplying two numbers using the standard multiplication algorithm. The two simpler problems are 345×8 and 345×40. What are the two numbers being multiplied?
9. In the problem $3456 \div b$, a student computed the partial quotients 400, 30, 60, and 3 and the remainder 5.
   a. Find the divisor.
   b. Write an equation involving division.
   c. Check your work.
10. Use properties to multiply 38 and 400.

plication facts. The digits are added along the diagonals, with regrouping as necessary, because each diagonal represents a place value. The digits on the left and bottom edges of the lattice reveal the product: $785 \times 47 = 36{,}895$.

	7	8	5	×
3	2/8	3/2	2/0	4
6	4/9	5/6	3/5	7
	8	9	5	

a. Use the given lattice to determine 456×82.

	4	5	6	×
3	3/2	4/0	4/8	8
7	0/8	1/0	1/2	2
	3	9	2	

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Where Are We Going?

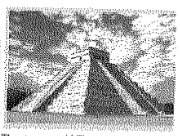
In **Section 10.1**, we introduce geometric language, diagrams, and symbolic notation that students need to describe, compare, and contrast geometric shapes and their properties. We also discuss point and line relationships. In **Section 10.2**, we discuss various angle relationships: vertical angles, complementary angles, supplementary angles, and angles created by a transversal. We also discuss attributes of triangles and parallelograms. In **Section 10.3**, we categorize objects in three dimensions.

How Did Ancient Cultures Use Geometry?

Geometry is the study of shapes and their patterns and properties. Ancient cultures used geometry in different ways according to their needs. The ancient Chinese used geometry to calculate heights and distances of objects in land surveys and to construct buildings and canals. The Mayans applied geometry in the construction of pyramids, temples, terraces, and reservoirs, as well as in their spiritual thinking. Geometry appears in their fabulous symmetric mosaic designs on ceramics and weavings. The ancient Egyptians used geometry to survey land and restore property boundaries after periodic flooding of the Nile and to construct religious temples and public buildings. They also cared about mathematical relationships between geometric figures, such as placing two right triangles together to form a rectangle, but some of their rules were inexact, such as the rule for the area of a circle ($A = 3.11r^2$).

What Is the Greek Influence on Geometry?

Although ancient cultures used geometry for religious, educational, and practical purposes, geometry progressed as an unorganized collection of results. The ancient Greeks, who learned geometry from the Egyptians, changed that by adding structure and logical reasoning, making geometry a subject to study from a mathematical and an abstract point of view. The word *geometry* stems from the Greek word *geometria*, which means "Earth measure." The Greeks' remarkable discoveries, methodical inquiry, and scholarly books led to the understanding and development of geometry in a systematic and organized way.

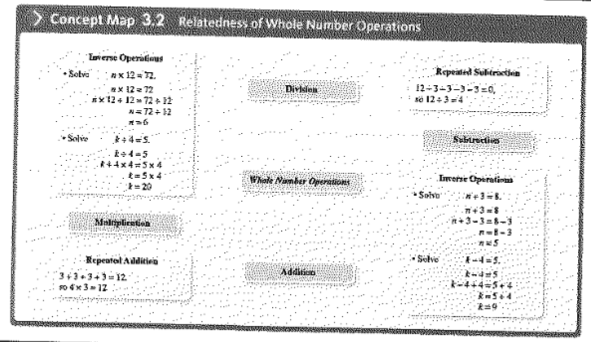


The step pyramid El Castillo ("the Castle"), built by the Mayans between the 900s and the 1300s.

In addition to the integration of the content and process standards, the text includes many features designed to engage students and prepare them to become elementary school teachers.

The Where Are We Going? **Chapter Opener** motivates the material in the chapter with inviting and brief questions (and their answers). This direct approach will help students navigate the main concepts of the chapter.

Unique **Concept Maps** engage students by showing how concepts relate to one another. The Concept Maps are rich with information and recap ideas in an accessible way.



The following Released Items illustrates other situations involving equally likely outcomes.

RELEASED ITEM

• NAEP, 2009

Marty has 6 red pencils, 4 green pencils, and 5 blue pencils. If he picks out one pencil without looking, what is the probability that the pencil will be green?

a. 1 out of 3 b. 1 out of 4 c. 1 out of 15 d. 4 out of 15

38% of fourth graders gave the correct answer.

• NAEP, 2007

(1, 1) (2, 1) (3, 1)
(1, 2) (2, 2) (3, 2)
(1, 3) (2, 3) (3, 3)

A pair of numbers will be chosen at random from the list above. What is the probability that the first number in the pair will be less than the second number in the pair?

33% of eighth graders gave the correct answer. (U.S. Department of Education,

The National Center for Education Statistics administers a nationwide standardized test—the National Assessment of Educational Progress (NAEP), also known as the Nation's Report Card—to a pooled random sample of 4th, 8th, and 12th graders. The NAEP released 25 and 29 questions from 4th- and 8th-grade tests in 1996, respectively; 59 and 67 questions from 4th- and 8th-grade tests in 2003, respectively; 32 and 56 questions from 4th- and 8th-grade tests in 2005, respectively; 54 and 53 questions from 4th- and 8th-grade tests in 2007, respectively; 31 and 34 questions from 4th- and 8th-grade tests in 2009, respectively; and 51 and 47 questions from 4th- and 8th-grade tests in 2011, respectively.

Some NAEP test questions given to 4th- and 8th-grade students appear in this textbook. The NAEP questions are commonly called Released Items because these questions were selectively released to improve student learning. The **Released Items** in the text show prospective teachers types of questions that elementary school students see in standardized assessment tests. The percentage of students who answered the question correctly is given.

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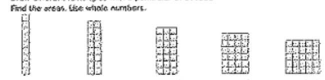
In the following Classroom Connection, students use long division or a calculator to classify rational numbers according to their decimal representation.

> **Classroom Connection**
>
> Harcourt *Math*, Student Edition, Grade 6, p. 247
>
> Rename each fraction as a terminating or repeating decimal. Write terminating or repeating. $\frac{11}{25}$, $\frac{4}{15}$. (© by Harcourt, Inc. Reproduced by permission of the publisher, Houghton Mifflin Harcourt Publishing Company.)

Some homework questions at the end of this section are designed to help you think about "exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters" (Gr. 3, CCSS). The following student page is an example of an activity that relates the perimeter and area of a rectangle.

> LESSON 4 HANDS ON
>
> **Relate Perimeter and Area**
>
> **Quick Review**
>
> Find the area of each rectangle.
>
> 1. 3 in. × 4 in.
> 2. 2 ft × 8 ft
> 3. 6 cm × 6 cm
> 4. 9 in. × 5 in.
> 5. 7 m × 11 m
>
> **MATERIALS** grid paper
>
> **Explore**
>
> Sharon and Brad are planning a rectangular sandbox for the preschool class. They have 20 feet of wood to make its sides. If they want their sandbox to have the greatest possible area, what dimensions should they use?
>
> You can use grid paper to model the relationship between perimeter and area. Find the greatest area of a rectangle with a perimeter of 20 feet.
>
> Draw different rectangles with a perimeter of 20 feet. Find the areas. Use whole numbers.



> | | | | | |
> |---|---|---|---|---|
> | $P = 20$ ft | $P = 20$ ft | $P = 20$ ft | $P = 20$ ft | $P = 20$ ft |
> | $A = l \times w$ | $A = l \times w$ | $A = l \times w$ | $A = l \times w$ | $A = l \times w$ |
> | $A = 1 \times 9$ | $A = 2 \times 8$ | $A = 3 \times 7$ | $A = 4 \times 6$ | $A = 5 \times 5$ |
> | $A = 9$ ft^2 | $A = 16$ ft^2 | $A = 21$ ft^2 | $A = 24$ ft^2 | $A = 25$ ft^2 |
>
> So, Sharon and Brad should build a 5-ft × 5-ft sandbox.
>
> - What dimensions gave the least area? What kind of rectangle gave the greatest area?

> **Try It**
>
> Use grid paper to draw rectangles for the given perimeter. Name the length and width of the rectangle with the greatest area. (Use whole numbers.)
>
> a. $P = 30$ cm b. $P = 24$ cm
>
> c. $P = 10$ cm d. $P = 38$ cm
>
> 530 CALIFORNIA STANDARDS

(Harcourt *Math*, Student Edition, Grade 6, p. 530)

Pre-K–8 problems from actual elementary mathematics textbooks appear in the exposition as **Classroom Connection** boxes. They give prospective teachers another opportunity to see that the topics they are studying are relevant to the mathematics curriculum. Furthermore, relevant pages from the actual elementary mathematics textbooks are featured. Each Classroom Connection box is placed in context so that prospective teachers can understand how textbooks for elementary school students present the information that they are currently studying in a college classroom.

This textbook includes integrated research results from publications and from Pre-K–8 classroom teachers to develop and clarify mathematical topics. Research results cannot prescribe the "best way" to teach a topic, but they do "show what is possible and what looks promising" (Heibert, 2003). They help determine reasonable approaches for presenting material and increase a prospective teacher's level of confidence in instructional decisions.

Key learning outcomes and ideas of the chapter are summarized in a chapter organizer grid. The **Chapter Organizer** presents the main elements of the chapter in a concise way so that they are accessible to students.

CHAPTER 2 Organizer

Section	What You Should Learn	Review Problems
2.1	1. Express ideas about collections of objects using vocabulary and notation.	1–3
	2. Represent sets with a Venn diagram.	4–6
	3. Find the union, intersection, set difference, and Cartesian product of two given sets.	7
	4. Use a Venn diagram to visualize sets and relationships.	8–12
2.2	1. Identify basic properties of a numeration system.	13–14
	2. Demonstrate understanding of place value concepts.	15–21
	3. Represent a base ten numeral in various forms.	22–24

Key Terms and Concepts

set 58	period 72	compatible numbers 91
element 58	value 72	decomposition 91
list form 58	standard form 74	identity for addition 92
universal set 58	word form 74	take away model 93
empty set 58	short word form 74	definition of subtraction 94
equal 58	expanded form 74	difference 94
equivalent 59	rounding 75	minuend 94
cardinality 59	Mayan numeration system 77	subtrahend 94
subset 61	Mayan place values 77	missing addend 94
proper subset 61	Babylonian numeration system 78	unknown addend 94
union 62	Babylonian place values 79	addend model 95
intersection 62	base five numeration system 79	comparison model 96
disjoint 62	base twelve numeration system 81	number line model (−) 97

Review Questions

1. Determine the type of variable.
   a. number of students who ride a bike to school
   b. student identification numbers
   c. letter grade on an assignment
   d. telephone numbers
   e. percentage of students who say their favorite color is blue
   f. age group of survey respondents: 13 to 18, 19 to 25, 26 to 30, 31 to 35, . . .
   g. types of crime (burglary, robbery, and so on)
2. The table shows data for five participants in a survey at an elementary school.

Student	ID	Favorite color	Pets
Janet	415	blue	4
Carlo	213	green	3
Mikey	004	blue	4
Elisa	681	brown	5
Pete	400	purple	0

   a. How many surveyed females were between 46 and 60 years old?
   b. How many males were surveyed?
   c. How many people were surveyed?
   d. What percentage of those surveyed were females?
5. *Consumer Reports* listed the prices of cell phones recommended for its subscribers. The manufacturer, brand of phone, and prices are listed.

Samsung; Impression; $50	Samsung; Solstice; $30
Motorola; Tundra; $180	LG; Rumor Touch; $30
Samsung; Instinct; $100	Samsung; Exclaim; $50
LG; Lotus Elite; $100	Samsung; Alias 2; $50
LG; enV Touch; $80	Casio; G'zOne Rock; $150
LG; enV3; $30	Samsung; Jitterbug J; $150
Samsung; Convoy; $70	Casio; G'zOne Brigade; $250

Display the data in a frequency table. Create categories that depend on the

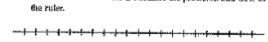
A variety of problems are given in a set of **Review Questions** to increase interest and flexibility in teaching the topics in this textbook.

The **Chapter Test** provides students an opportunity to reinforce their learning—even as the display of learning outcomes in the chapter tests offers students an opportunity to consider how knowledge acquisition works. The chapter tests are also designed to support the national trend of integrating instruction, assessment, and content and process standards.

Chapter 11 Test

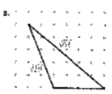
1. *Understand major principles of measurement.* List two attributes of a jar that you can measure.
2. *Understand major principles of measurement.*

   a. Explain how an elementary school student could mistakenly report the length of the nail as $4\frac{1}{8}$ inches.
   b. Determine the correct length of the nail. Show your work.
3. *Determine the precision and greatest possible error (GPE).* Part of a ruler is shown. Determine the precision and GPE of the ruler.

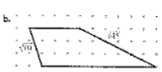


7. *Determine the perimeter or area of a polygon.* Use formulas to find the area of each polygon.

a.



b.



Other Key Features

Historical Notes provide social, cultural, and historical context for select mathematical ideas. They remind students that mathematics is a human activity and an evolving process.

Important **Definitions and Key Theorems and Properties** are set in two types of boxes designed to distinguish them. Definitions briefly describe key terms to learn.

Key terms appear in boldface type to make them easier to locate.

Historical Note

William Playfair (1759–1823), a draftsman and engineer, invented line plots, bar graphs, pie charts, and histograms, publishing them in the books *The Commercial and Political Atlas* and *Statistical Breviary*. In the preface of his atlas, Playfair makes the case that he was the first to represent data with picture graphs (also called pictographs).

Regarding tables, Playfair said, "A man who has carefully investigated a printed table, finds, when done, that he has only a very faint and partial idea of what he has read, and that like a figure imprinted on sand, is soon totally erased and defaced" (*Atlas*, 1801, p. xiv). Regarding graphs, he said, "as much information may be obtained in five minutes as would require whole days to imprint on the memory, in a lasting manner, by a table of figures" (*Atlas*, p. xii) and "Of all the senses, the eye gives the liveliest and most sensible idea of whatever is susceptible of being represented to it" (*Statistical Breviary*, 1801, 14).

Initially, scientists viewed picture representations with suspicion, preferring numbers and formal reasoning instead. Graphs were difficult to publish at that time because of the difficulty of engraving illustrations on copper plates. With improved printing technology and increased acceptance of inductive reasoning, the graphs gradually became acceptable tools for representing data and making reasonable inferences. Graphs today still model Playfair's use of descriptive titles, frames, shading, grid lines, and labels for axes.

Theorem

Let a and b represent any whole numbers. Then $a \cdot b = \text{LCM}(a, b) \cdot \text{GCF}(a, b)$.

Noncomputational Definition of $a \times b$

Let the counting number a represent the number of groups and b represent the number of objects in each group. Then the total number of objects in all groups is $a \times b$, which is read as "a times b."

- a is called the **multiplier**, and b is called the **multiplicand.**
- a and b are also called **factors.**
- The number $a \times b$ is also read as "**the product of a and b.**"
- We define $0 \times b$ as equal to zero ($0 \times b = 0$).

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A Brief Overview

This textbook:

- focuses on the goal of giving prospective teachers a lasting understanding of the factual, conceptual, and procedural knowledge that they are expected to know and be able to teach.
- includes problems from actual elementary mathematics textbooks, Released Items from the NAEP, and selected NCTM Standards and Expectations and CCSS that link the mathematics prospective teachers study to the mathematics they will teach, as appropriate. This combination helps students answer the questions "What do we need to know?" and "Why do we need to know it?" This combination makes mathematics more relevant for prospective teachers, which will increase their persistence, improve their attitude, and reduce their anxiety. This allows the mathematics instructor to focus on the mathematical content while teaching the course.
- promotes the five processes of mathematics (representation, problem solving, reasoning and proof, connections, and communication). The processes provide a bridge to teaching and learning mathematical content. The processes that a student uses reveal much about the student's level of understanding—which is important for planning, instruction, and assessment.
- contains a variety of homework problems at the end of each section and chapter to increase a student's interest and flexibility in teaching the topics from the textbook. The problems reflect the content and processes that prospective teachers must know and teach. They also reflect the national trend of integrating instruction, assessment, and content and process standards. Although the five processes underlie all questions in the homework, some questions are organized in groups to give prospective teachers a chance to explore a particular process in more depth and to improve their abilities in selecting problems that assess the process. This unique differentiation allows the mathematics instructor to assess a particular aspect of student understanding of a mathematical topic.
- views procedural understanding (*how*) as the culmination of conceptual understanding (*why*), which improves motivation and attitude. The chapter tests found at the end of the chapters reinforce and reconsider these hows and whys by linking problems to specific learning outcomes.
- reflects the belief that prospective teachers should know the content at a deeper level than they teach because they will lead classroom discussions, ask their students questions, and answer questions from their students. The Math Panel Report (2008) supports this assertion: "teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connection of that content to other important mathematics, both prior to and beyond the level they are assigned to teach" (p. 38).

Supplements for the Instructor

Print Supplements

Instructor Edition
(ISBN: 978-1-111-98946-0)
The *Instructor Edition* features an appendix containing the answers to all problems in the book, as well as an appendix denoting which problems can be found in Enhanced WebAssign®.

Instructor's Resource Manual
(ISBN: 978-1-133-36372-9)
Authors: Ricardo D. Fierro and Scott Fallstrom
The *Instructor's Resource Manual* provides detailed solutions to all problems in the text.